

# DEGENERATIONS OF GODEAUX SURFACES AND EXCEPTIONAL VECTOR BUNDLES

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**ABSTRACT.** A recent construction of Hacking relates the classification of stable vector bundles on a surface of general type with  $p_g = 0$  and the boundary of the moduli space of deformations of the surface. In the present paper we analyze this relation for Godeaux surfaces. We provide a description of certain boundary components of the moduli space of Godeaux surfaces. Also we explicitly construct certain exceptional vector bundles of rank 2 on Godeaux surfaces, stable with respect to the canonical class, and examine the correspondence between the boundary components and such exceptional vector bundles.

## 1. INTRODUCTION

Complex algebraic surfaces  $Y$  of Kodaira dimension 2 are called surfaces of general type. Such surfaces are classified according to discrete topological invariants  $K^2 = c_1(Y)^2$  and  $\chi = \chi(\mathcal{O}_Y)$ . Having fixed these invariants, one can consider the space of all surfaces of general type with given invariants, i.e., the moduli space  $\mathcal{M} = \mathcal{M}_{K^2, \chi}$ , which itself has the structure of an algebraic variety.

While not compact,  $\mathcal{M}_{K^2, \chi}$  admits a natural compactification, the moduli space  $\overline{\mathcal{M}}_{K^2, \chi}$  of stable surfaces introduced by Kollár and Shepherd-Barron [KSB88] and Alexeev [Ale96]. This is an analog of the compactification of Deligne and Mumford  $\overline{\mathcal{M}}_g$ , of the moduli space  $\mathcal{M}_g$  of smooth curves of genus  $g \geq 2$  [DM69].

Such stable surfaces can contain isolated singularities or even mild singularities along curves. Also the moduli space  $\overline{\mathcal{M}}_{K^2, \chi}$  can be arbitrarily singular [Vak06], or have arbitrarily many connected components [Cat86]. Fortunately, some connected components of the boundary of  $\overline{\mathcal{M}}_{K^2, \chi}$  are relatively well-behaved. They correspond to a degeneration of a smooth surface of general type to a surface with a unique quotient singularity of a special type, first studied by Wahl [Wah81].

**Definition 1.1.** A *singularity of Wahl type*  $\frac{1}{n^2}(1, na - 1)$  is a cyclic quotient singularity  $0 \in (\mathbb{C}^2/\mathbb{Z}/n^2\mathbb{Z})$  given by

$$\mathbb{Z}/n^2\mathbb{Z} \ni 1 : (u, v) \mapsto (\zeta u, \zeta^{na-1}v),$$

where  $a$  and  $n$  are positive integers such that  $a < n$ ,  $(a, n) = 1$ , and  $\zeta = \exp(2\pi i/n^2)$ .

In the absence of local-to-global obstructions, a singularity of Wahl type on a surface  $X$  admits a smoothing  $Y \rightsquigarrow X$  such that  $H_2(Y, \mathbb{Q}) \simeq H_2(X, \mathbb{Q})$ . The local smoothing is determined by a single deformation parameter, so the locus of equisingular deformation of  $X$  defines a codimension 1 boundary component of the moduli space of deformations of  $Y$ . We will call such a boundary component a  $\frac{1}{n^2}(1, na - 1)$  boundary component.

In this paper we investigate some of the boundary components of the moduli space  $\overline{\mathcal{M}}_{1,1}$ , containing Godeaux surfaces. A Godeaux surface is a minimal surface of general type with invariants  $K^2 = 1$  and  $p_g = 0$ . Godeaux surfaces are in some sense the simplest surfaces of general type satisfying  $H^1(Y) = H^{2,0}(Y) = 0$ .

In [Hac13a], Hacking describes a way to construct exceptional vector bundles of rank  $n$  on a smooth surface  $Y$  such that  $H^1(Y) = H^{2,0}(Y) = 0$  using degenerations  $Y \rightsquigarrow X$  of  $Y$  to a surface  $X$  with a unique singularity of Wahl type  $\frac{1}{n^2}(1, na - 1)$ . Exceptional vector bundles on a surface  $Y$ , discussed in Section 3, are holomorphic vector bundles  $E$  such that  $\text{Hom}(E, E) = \mathbb{C}$  and  $\text{Ext}^1(E, E) = \text{Ext}^2(E, E) = 0$ . In particular, such a vector bundle is indecomposable, rigid and unobstructed in families, i.e., it deforms in a unique way in a family of surfaces. Exceptional vector bundles have also appeared in decompositions of the derived categories on Godeaux surfaces [BBS13] and on Burniat surfaces [AO13].

The construction of Hacking gives rise to a correspondence

$$(1) \quad \{\text{Wahl degenerations}\} \longrightarrow \{\text{exceptional vector bundles}\} / \sim,$$

where  $\sim$  denotes the equivalence relation defined in Remark 3.8.

This correspondence is bijective in the case  $Y = \mathbb{P}^2$  [Hac13a]. Thus it was natural to study the correspondence (1) for other surfaces, in particular surfaces of general type. In the present paper we examine the correspondence (1) for Godeaux surfaces and Wahl degenerations corresponding to the  $\frac{1}{4}(1, 1)$  singularity.

Throughout the paper we will make use of the classification of Godeaux surfaces according to  $H_1(Y, \mathbb{Z})$ , which is cyclic of order at most 5 [Rei78]. There exists a complete description of the moduli space of Godeaux surfaces in each of the cases  $H_1(Y, \mathbb{Z}) = \mathbb{Z}/5\mathbb{Z}$ ,  $\mathbb{Z}/4\mathbb{Z}$  or  $\mathbb{Z}/3\mathbb{Z}$  [Rei78]. Interestingly, for the remaining two cases, the description of the moduli space of Godeaux surfaces is still unknown. Several examples of Godeaux surfaces with  $H_1(Y, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$  were constructed in [Bar84], [KLP10], [CD89] and some work towards classification is done in [C09]. In the case  $H_1(Y, \mathbb{Z}) = 0$  the only known examples are described in [Bar85], [DW99], [LP07], but it is not even known if these examples belong to the same irreducible component of the moduli space.

**1.1. Results.** First, we classify all possible degenerations of a smooth Godeaux surface  $Y$  to a surface  $X$  with unique singularity of Wahl type  $\frac{1}{4}(1, 1)$ , such that  $K_X$  is ample. In other words, we describe the boundary components of the KSBA compactification of the moduli space of smooth Godeaux surfaces corresponding to surfaces with a unique such singularity.

**Theorem 1.2.** *The  $\frac{1}{4}(1, 1)$  boundary components in the the KSBA compactification all parametrize surfaces whose minimal resolution is a proper elliptic surface. There are no such components when  $H_1(Y, \mathbb{Z}) = \mathbb{Z}/5\mathbb{Z}$ , and at least one for each other possible value of  $H_1(Y, \mathbb{Z})$ .*

See Theorem 2.3 for a more precise statement.

Second, we classify all exceptional vector bundles of rank 2 on smooth Godeaux surfaces into two orbits under the natural equivalence relation obtained from the construction of Hacking. We provide the complete description for one of the orbits.

**Theorem 1.3.** *If  $E$  is a  $K_Y$ -stable exceptional vector bundle of rank 2 on a Godeaux surface  $Y$  with  $c_1(E) = K_Y$  modulo torsion, then, after tensoring by a torsion line bundle,  $E$  can be written as an extension*

$$0 \rightarrow \mathcal{O}_Y \rightarrow E \rightarrow \mathcal{O}_Y(K_Y + \sigma) \otimes \mathcal{I}_P \rightarrow 0,$$

where  $\mathcal{I}_P$  is the ideal sheaf of a point  $P$  which is a base point of  $|2K_Y + \sigma|$ , and  $\sigma \in \text{Tors } Y \setminus 2\text{Tors } Y$ , so we must have  $H_1(Y) = \mathbb{Z}/4\mathbb{Z}$  or  $\mathbb{Z}/2\mathbb{Z}$ .

*Conversely, given  $P$  and  $\sigma$  as above, there is a unique non-trivial extension  $E$  of this form, and  $E$  is a  $K_Y$ -stable exceptional vector bundle provided  $P$  is a simple basepoint.*

Finally, we investigate which of these exceptional vector bundles can be obtained using Hacking's construction.

**Theorem 1.4.** *Let  $Y$  be a Godeaux surface with  $H_1(Y, \mathbb{Z}) = \mathbb{Z}/4\mathbb{Z}$ . Every  $K_Y$ -stable exceptional bundle  $E$  of rank 2 on  $Y$  with  $c_1(E) = K_Y$  modulo torsion is equivalent to one arising from a  $\frac{1}{4}(1, 1)$  Wahl degeneration  $Y \rightsquigarrow X$  with  $K_X$  ample.*

The paper is structured as follows. Section 2 deals with the classification of the degenerations of a Godeaux surface  $Y$  to a surface  $X$  with a unique singularity of Wahl type  $\frac{1}{4}(1, 1)$ , such that the canonical divisor  $K_X$  is ample. A complete classification is provided and some concrete examples are described in detail.

Section 3 contains the analysis of the equivalence classes of certain exceptional vector bundles of rank 2 on smooth Godeaux surfaces  $Y$ . The classification is provided modulo the equivalence relation arising from the construction of Hacking. A complete description is provided for one of the two equivalence classes.

Section 4 analyzes the correspondence (1) in the case  $Y$  is a Godeaux surface and  $X$  has a unique singularity of the type  $\frac{1}{4}(1, 1)$ .

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#### 2. WAHL DEGENERATIONS

We use the following notation. Let  $Y$  be a Godeaux surface, and let  $X$  be a  $\mathbb{Q}$ -Gorenstein degeneration of  $Y$  such that  $X$  has a unique singularity, which is of Wahl type  $\frac{1}{4}(1, 1)$  and  $K_X$  is nef. We denote by  $\tilde{X}$  be the minimal resolution of  $X$ . For a point  $P$  on  $X$  we write  $(P \in X)$  to denote a small complex analytic neighborhood of  $P \in X$ . We use the notation  $\text{Tors}(Y) = \text{Tors } H^2(Y, \mathbb{Z}) = \text{Tors } H^2(Y) = \text{Tors } H_1(Y) = H_1(Y)$ .

**Proposition 2.1.** *The surface  $\tilde{X}$  is a minimal properly elliptic surface, i.e.  $\tilde{X}$  is minimal of Kodaira dimension 1.*

*Proof.* We start by computing invariants of the surface  $\tilde{X}$ . Notice that since  $X$  is a  $\mathbb{Q}$ -Gorenstein degeneration of  $Y$ , we have  $K_X^2 = K_Y^2 = 1$ . The exceptional locus  $C$  of the minimal resolution of the cyclic quotient singularity of type  $\frac{1}{4}(1, 1)$  consists of a single  $(-4)$  curve.

Let  $\pi : \tilde{X} \rightarrow X$ . Then  $K_{\tilde{X}} = \pi^* K_X - \frac{1}{2}C$  by the adjunction formula, so  $K_{\tilde{X}}^2 = 0$ .

Also  $p_g(\tilde{X}) = h^2(\mathcal{O}_{\tilde{X}}) = h^2(\mathcal{O}_X)$  since  $(P \in X)$  is a rational singularity, and  $h^2(\mathcal{O}_X) = h^2(\mathcal{O}_Y) = 0$  since  $(P \in X)$  is a quotient singularity by [DB81], 4.6 and 5.3. Similarly  $q(\mathcal{O}_{\tilde{X}}) = h^1(\mathcal{O}_{\tilde{X}}) = h^1(\mathcal{O}_Y) = 0$ . Finally, the fundamental group does not change on a resolution of a rational singularity, so  $\pi_1(\tilde{X}) = \pi_1(X)$ .

So  $\tilde{X}$  is a surface with  $K_{\tilde{X}}^2 = 0$ ,  $p_g(\tilde{X}) = q(\tilde{X}) = 0$ . By Lemma 2.2 below  $\tilde{X}$  is a minimal surface. Thus according to the classification of minimal surfaces [BHPV04,

p. 244], the surface  $\tilde{X}$  is either an Enriques surface, a K3 surface, or a properly elliptic surface. Note that  $\tilde{X}$  cannot be an Enriques surface or a K3 surface because there exists a curve  $C$  on  $\tilde{X}$  such that  $C \cdot K_{\tilde{X}} = 2$ , thus  $2K_{\tilde{X}}$  and  $K_{\tilde{X}}$  are not numerically trivial. Therefore the surface  $\tilde{X}$  is a properly elliptic surface.  $\square$

**Lemma 2.2.** *The surface  $\tilde{X}$  does not contain a  $(-1)$ -curve. Thus  $\tilde{X}$  is not a rational surface or a surface of general type.*

*Proof.* Arguing by contradiction, suppose that there exists a  $(-1)$ -curve  $F$  on  $\tilde{X}$ .

Denote by  $\bar{F} = \pi(F)$  the image of the curve  $F$  under the map  $\pi$ . Recall  $K_{\tilde{X}} = \pi^*(K_X) - \frac{1}{2}C$ . Since the canonical divisor  $K_X$  is big and nef, its intersection index with the curve  $\bar{F} = \pi_*(F)$  on  $X$  must be nonnegative. But

$$K_X \cdot \bar{F} = K_X \cdot \pi_*(F) = \pi^*(K_X) \cdot F = (K_{\tilde{X}} + \frac{1}{2}C) \cdot F = -1 + \frac{1}{2}C \cdot F \geq 0,$$

thus  $C \cdot F \geq 2$ .

Now consider the divisor  $D = C + 2F$  on  $\tilde{X}$ . We have  $D^2 = -8 + 4C \cdot F \geq 0$  since  $C \cdot F \geq 2$ . But  $K_{\tilde{X}} \cdot D = 0$ , so  $D \in K_{\tilde{X}}^\perp$  and  $K_{\tilde{X}}^2 = 0$ . Note that  $D$  and  $K_{\tilde{X}}$  are linearly independent in  $H^2(\tilde{X}, \mathbb{R})$ . If  $D^2 > 0$  it is clear that  $D$  and  $K_{\tilde{X}}$  are linearly independent because  $K_{\tilde{X}}^2 = 0$ . If  $D^2 = 0$  then  $C \cdot F = 2$  and we find  $D \cdot F = 0$ ,  $K_{\tilde{X}} \cdot F = -1$ , so again  $D$  and  $K_{\tilde{X}}$  are linearly independent. This contradicts the Hodge Index Theorem. Therefore there is no such curve  $F$ .  $\square$

**Theorem 2.3.** *The surface  $\tilde{X}$  is a properly elliptic surface over  $\mathbb{P}^1$  with two multiple fibers of multiplicities  $m_1, m_2$ . In particular  $\pi_1(\tilde{X}) \simeq \mathbb{Z}/(m_1, m_2)\mathbb{Z}$ . Write  $n := C \cdot A$ , where  $A$  is a general fiber of the elliptic fibration and  $C$  is the exceptional locus of  $\pi : \tilde{X} \rightarrow X$ , then we have the following possibilities for  $m_1, m_2$  and  $n$ :*

- (a)  $m_1 = 4, m_2 = 4, n = 4$ ;
- (b)  $m_1 = 3, m_2 = 3, n = 6$ ;
- (c)  $m_1 = 2, m_2 = 6, n = 6$ ;
- (d)  $m_1 = 2, m_2 = 4, n = 8$ ;
- (e)  $m_1 = 2, m_2 = 3, n = 12$ .

*Proof.* We have an elliptic fibration

$$\begin{array}{ccc} \tilde{X} & \longleftarrow & \{\text{multiple fibers with multiplicities } m_i\} \\ f \downarrow & & \downarrow \\ B & \longleftarrow & \{P_i\} \end{array}$$

Denote by  $L$  the dual of the line bundle  $R^1 f_* \mathcal{O}_{\tilde{X}}$  on  $B$ . By [FM94], Chapter I, Lemma 3.18 we have  $\deg L = \chi(\mathcal{O}_{\tilde{X}}) = 1$ , since  $p_g(\tilde{X}) = q(\tilde{X}) = 0$ . Moreover, according to [FM94], Chapter I, Proposition 3.22, we can compute the genus  $g(B)$  of the base curve  $B$  using the relation  $p_g(\tilde{X}) = \deg L + g(B) - 1 = 0$ . Thus the base curve  $B$  must be isomorphic to  $\mathbb{P}^1$ .

Since the Euler number  $e(\tilde{X}) > 0$ , we note that by [FM94], Chapter II, Theorem 2.3 the fundamental group  $\pi_1(\tilde{X})$  is isomorphic to the orbifold fundamental

group of the base  $B \simeq \mathbb{P}^1$ , i.e., we have an isomorphism:

$$\pi_1(\tilde{X}) \simeq \frac{\pi_1(B \setminus \{P_1, \dots, P_r\})}{\langle \gamma_1^{m_1}, \dots, \gamma_r^{m_r} \rangle},$$

where  $\gamma_i$  are loops on the base  $B$  of the fibration about the corresponding points  $P_i$ .

Now we consider the Kodaira canonical bundle formula ([Fri98], Theorem 15). Let  $f : \tilde{X} \rightarrow B$  be a relatively minimal elliptic fibration. Suppose that  $F_1, \dots, F_k$  are the multiple fibers of  $f$  and that the multiplicity of  $F_i$  is  $m_i$ . Then:

$$\omega_{\tilde{X}} = f^*(\omega_B \otimes L) \otimes \mathcal{O}_{\tilde{X}}\left(\sum_i (m_i - 1)F_i\right).$$

Thus we obtain the following formula for the canonical line bundle of  $\tilde{X}$ .

$$(2) \quad K_{\tilde{X}} = \left(-1 + \sum_i \left(\frac{m_i - 1}{m_i}\right)\right) A$$

in  $\text{Pic}(\tilde{X}) \otimes \mathbb{Q} \simeq H^2(\tilde{X}, \mathbb{Q})$ , where  $A$  is a general fiber of the fibration.

Recall that we have a  $(-4)$ -curve  $C$  on  $\tilde{X}$ , and that  $C \cdot K_{\tilde{X}} = 2$ .

Intersecting (2) with  $C$  we obtain the equation  $2 = (-1 + \sum \frac{m_i - 1}{m_i})n$ , or

$$(3) \quad 1 + \frac{2}{n} = \sum_i \left(\frac{m_i - 1}{m_i}\right).$$

Since  $n = C \cdot A = (C \cdot F_i)m_i$ , we note that  $m_i$  divides  $n$ . Given these conditions, the only integer solutions of the equation (3) are:

- (a)  $m_1 = 4, m_2 = 4, n = 4$ ;
- (b)  $m_1 = 3, m_2 = 3, n = 6$ ;
- (c)  $m_1 = 2, m_2 = 6, n = 6$ ;
- (d)  $m_1 = 2, m_2 = 4, n = 8$ ;
- (e)  $m_1 = 2, m_2 = 3, n = 12$ ;
- (f)  $m_1 = m_2 = m_3 = 2, n = 4$ ;
- (g)  $m_1 = m_2 = m_3 = m_4 = 2, n = 2$ .

Thus  $\pi_1(\tilde{X})$  is Abelian in all cases except the last two. We have  $H_1(\tilde{X}) = \pi_1^{ab}(\tilde{X})$  so in the case (f) we can compute  $H_1(\tilde{X}) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ , and in the case (g) we have  $H_1(\tilde{X}) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ . According to [Hac13b], p.134 we have a surjection  $\phi : H_1(Y) \twoheadrightarrow H_1(X)$  and the kernel of the map  $\phi$  is either  $\mathbb{Z}/2\mathbb{Z}$  or trivial. (Here  $n = 2$  for the singularity of Wahl type  $\frac{1}{n^2}(1, na - 1)$ .) Thus the last two cases cannot be obtained by a degeneration of a Godeaux surface  $Y$  to a surface  $X$ .  $\square$

**Remark 2.4.** Since we have a map  $\phi : H_1(Y) \rightarrow H_1(X)$ , which kernel is either  $\mathbb{Z}/2\mathbb{Z}$  or trivial,  $H_1(Y) \simeq H_1(X)$  in the cases  $H_1(Y) = \mathbb{Z}/5\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}$ , or trivial. We can have  $\mathbb{Z}/2\mathbb{Z} \rightarrow H_1(Y) \rightarrow H_1(X)$  in the remaining two cases. However, the construction of Hacking requires  $H_1(Y) \simeq H_1(X)$ , so we will provide specific examples for which this condition is satisfied.

The following statement is an immediate corollary of Theorem 2.3.

**Corollary 2.5.** *There does not exist a degeneration of a smooth Godeaux surface  $Y$  with  $H_1(Y) = \mathbb{Z}/5\mathbb{Z}$  to a surface  $X$  with a unique singularity ( $P \in X$ ) of Wahl type  $\frac{1}{4}(1, 1)$ , such that  $K_X$  is ample.*

We provide explicit constructions of degenerations  $Y \rightsquigarrow X$  when  $H_1(Y, \mathbb{Z})$  is not equal to  $\mathbb{Z}/5\mathbb{Z}$ .

**Proposition 2.6.** *There exists a  $\mathbb{Q}$ -Gorenstein degeneration of a Godeaux surface  $Y$  with  $H_1(Y) = \mathbb{Z}/4\mathbb{Z}$  to a surface  $X$  with a unique singularity of type  $\frac{1}{4}(1, 1)$  such that  $K_X$  is ample.*

*Proof.* According to [Rei78] the universal cover  $\bar{Y}$  of a Godeaux surface  $Y$  with  $H_1(Y) = \mathbb{Z}/4\mathbb{Z}$  is given by a complete intersection of two quartics  $\bar{Y} = \{q_0 = q_2 = 0\} \subset \mathbb{P}(1^3, 2^2)$  in a weighted projective space with coordinates  $x_1, x_2, x_3, y_1, y_3$  of degrees 1, 1, 1, 2, 2, respectively. Then the Godeaux surface  $Y$  is the quotient of  $\bar{Y}$  by the  $\mathbb{Z}/4\mathbb{Z}$  action generated by  $x_i \mapsto \zeta^{i-1}x_i$ ,  $y_i \mapsto \zeta^{i-2}y_i$ , where  $\zeta$  is a primitive 4th root of unity. We will say that a variable  $x_j$  has weight  $j \in \mathbb{Z}/4\mathbb{Z}$  to indicate that  $x_j \mapsto \zeta^j x_j$  under the  $\mathbb{Z}/4\mathbb{Z}$  action.

We consider the family  $\mathcal{P} = (x_1x_3 = x_2^2 + tv_0) \subset \mathbb{P}(1^3, 2^3) \times \mathbb{A}_t^1$  with coordinates  $x_1, x_2, x_3, v_0, y_1, y_3, t$  which have weights 0, 1, 2, 2, 3, 1, 0  $\in \mathbb{Z}/4\mathbb{Z}$  respectively. Then if  $t \neq 0$ , we can solve for  $v_0$  and so the general fiber  $\mathcal{P}_t$  is isomorphic to  $\mathbb{P}(1^3, 2^2)$ .

The special fiber  $\mathcal{P}_0$  is isomorphic to the projective space  $\mathbb{P}(1^2, 4^3)$  with coordinates  $u_0, u_1, v_0, y_1, y_3$  with weights 0, 1, 2, 3, 1  $\in \mathbb{Z}/4\mathbb{Z}$  via setting  $x_1 = u_0^2$ ,  $x_2 = u_0u_1$ ,  $x_3 = u_1^2$ . Then  $\bar{Y} \subset \mathcal{P}_t$  degenerates to a complete intersection  $\bar{X} \subset \mathbb{P}(1^2, 4^3)$ , given by two equations of degree 8 and weights 0 and 2.

We describe an example of a  $\mathbb{Z}/4\mathbb{Z}$  invariant quasismooth complete intersection in  $\mathbb{P}(1, 1, 4, 4, 4)$ . We define  $\bar{X} = \{f_0 = f_2 = 0\}$ , where

$$(4) \quad \begin{aligned} f_0 &= u_0^8 + u_1^8 + u_0^4 u_1^4 + y_1 y_3 + v_0^2 + v_0 u_0^2 u_1^2; \\ f_2 &= u_0^6 u_1^2 + u_0^2 u_1^6 + y_1^2 + y_3^2 + u_0^4 v_0 + u_1^4 v_0. \end{aligned}$$

Now  $\bar{X}$  meets the locus  $(u_0 = u_1 = 0) \subset \mathbb{P}(1^2, 4^3)$  transversely at exactly four points  $(0, 0, 1, \pm\zeta, \mp\zeta) \in \mathbb{P}(1^2, 4^3)$ , where  $\zeta^4 = -1$ . Thus  $\bar{X}$  has four  $\frac{1}{4}(1, 1)$  singularities and no other singularities, and its quotient  $X = \bar{X}/(\mathbb{Z}/4\mathbb{Z})$  has a unique  $\frac{1}{4}(1, 1)$  singularity.

Using the adjunction formula to compute the canonical divisor of  $X$  we obtain  $K_{\bar{X}} = (-14H + 8H + 8H)|_{\bar{X}} = 2H|_{\bar{X}}$ , where  $H$  is a general hyperplane divisor on  $\mathbb{P}(1^2, 4^3)$ , so  $K_X = p_* K_{\bar{X}}$  is ample.  $\square$

In the proof of Proposition 2.6, we constructed degeneration on the ambient weighted projective space. In contrast, we will now construct a degeneration in the  $H_1(Y, \mathbb{Z}) = \mathbb{Z}/3\mathbb{Z}$  by choosing the equations which meet the singular locus of the weighted projective space.

**Proposition 2.7.** *There exists a  $\mathbb{Q}$ -Gorenstein degeneration of a Godeaux surface  $Y$  with  $H_1(Y) = \mathbb{Z}/3\mathbb{Z}$  to a surface  $X$  with a unique singularity of type  $\frac{1}{4}(1, 1)$ , such that  $K_X$  is ample.*

*Proof.* In the weighted projective space  $\mathbb{P}(1^3, 2^3, 3^3)$  with coordinates  $x_i, y_i, z_i$ , where  $i \in \{0, 1, 2\}$ , consider equations

$$\begin{aligned}
 (5) \quad & r_2 x_1 x_0 - x_2 z_2 + y_1 y_0 = 0 \\
 & x_1 z_1 - x_2 r_1 x_0 - y_0 y_2 = 0 \\
 & x_1 (S x_0 - r_0 y_0) - y_1 r_1 x_0 - z_2 y_2 = 0 \\
 & x_2 (S x_0 - r_0 y_0) - y_1 z_1 - y_2 r_2 x_0 = 0 \\
 & y_0 (S x_0 - r_0 y_0) - z_1 z_2 + r_1 r_2 x_0^2 = 0 \\
 & x_0 z_0 - y_1 y_2 - r_0 x_1 x_2 = 0 \\
 & y_0 z_0 - S x_1 x_2 + r_2 x_1 y_2 + r_1 x_2 y_1 = 0 \\
 & z_0 z_1 - S x_2 y_2 - r_0 r_1 x_2^2 + r_2 y_2^2 = 0 \\
 & z_0 z_2 - S x_1 y_1 - r_0 r_2 x_1^2 + r_1 y_1^2 = 0
 \end{aligned}$$

Miles Reid showed in [Rei00] that any Godeaux surface  $Y$  with  $H_1(Y) = \mathbb{Z}/3\mathbb{Z}$  can be obtained by setting in (5)  $x_0 + x_1 + x_2 = 0$ ,  $z_0 + z_1 + z_2 = 0$ , and  $r_i =$  quadratic,  $S =$  cubic expression in  $x_i, y_i$ , with the  $\mathbb{Z}/3\mathbb{Z}$ -action given by cyclic permutation of  $(0, 1, 2)$ . See the example 7.1 in [Rei00] for details. (See also [Rei78], section 3.) Moreover, denote by  $W \subset \mathbb{P}(1^3, 2^3, 3^3)$  the  $\mathbb{Z}/3\mathbb{Z}$  cover of  $Y$ , then  $K_W = H|_W$ , where  $H$  is a general hyperplane on  $\mathbb{P}(1^3, 2^3, 3^3)$ .

We consider a one parameter family of Godeaux surfaces given by setting in the equations (5)

$$\begin{aligned}
 r_0 &= t y_0 + y_1 + 2 y_2 + x_1^2 \\
 r_1 &= t y_1 + y_2 + 2 y_0 + x_2^2 \\
 r_2 &= t y_2 + y_0 + 2 y_1 + x_0^2 \\
 S &= x_0^3 + x_1^3 + x_2^3,
 \end{aligned}$$

where  $t \in \mathbb{C}^1$ , along with  $x_0 + x_1 + x_2 = 0$  and  $z_0 + z_1 + z_2 = 0$ . Then for each fixed small  $t \neq 0$  after taking a quotient by the group action we obtain a smooth Godeaux surface with  $H_1 = \mathbb{Z}/3\mathbb{Z}$ .

Using Macaulay2, we can see that at  $t = 0$  the surface  $Z$  defined by these equations has three  $\frac{1}{4}(1, 1)$  singularities at the points where  $x_i = z_i = 0$ ,  $y_j = 1$ ,  $y_k = 0$  for all  $i$  and  $j \neq k \in \{0, 1, 2\}$  and no other singularities. Direct calculation shows that all three of these singularities are  $\frac{1}{4}(1, 1)$  singularities. Thus the quotient by the  $\mathbb{Z}/3\mathbb{Z}$  action is a surface with a unique singularity of type  $\frac{1}{4}(1, 1)$  which is a degeneration of a smooth Godeaux surface. Also  $K_Z = H|_Z$ , so  $K_X$  is ample.  $\square$

A construction of a degeneration  $X$  of a Godeaux surface  $Y$  with a unique singularity of type  $\frac{1}{4}(1, 1)$  in the cases  $H_1(Y) = \mathbb{Z}/2\mathbb{Z}$  and  $H_1(Y) = 0$  arises from the theory of  $\mathbb{Q}$ -Gorenstein smoothing for projective surfaces with special quotient singularities.

**Proposition 2.8.** *Let  $Y$  be a simply connected Godeaux surface, whose construction is described in the [LP07], Section 7, Construction A2. Then there is a degeneration  $Y \rightsquigarrow X$ , where  $X$  has a unique singularity of type  $\frac{1}{4}(1, 1)$ , and  $K_X$  is nef. Let  $\tilde{X}$  be a minimal resolution of  $X$ , then  $\tilde{X}$  is a Dolgachev surface, i.e. a properly elliptic surface over  $\mathbb{P}^1$  with exactly two multiple fibers of multiplicities 2 and 3.*

*Proof.* The surface  $X$  is obtained by smoothing all but one  $\frac{1}{4}(1, 1)$  singularity in the surface  $X'$  in the construction of Lee and Park. Lee and Park give an example of a simply connected Godeaux surface using  $\mathbb{Q}$ -Gorenstein smoothing theory [LP07]. They first construct a surface  $X'$  having one cyclic quotient singularity of type  $\frac{1}{36}(1, 5)$  and two cyclic quotient singularities of each of the types  $\frac{1}{4}(1, 1)$  and  $\frac{1}{16}(1, 3)$  such that  $K_{X'}$  is nef. They prove that  $X'$  has a  $\mathbb{Q}$ -Gorenstein smoothing such that a general fiber of the family is a simply connected, minimal, complex surface of general type with  $p_g = 0$  and  $K^2 = 1$ . See [LP07], Section 7, Construction A2 for details on the construction, as well as the outline of the proof.

In particular, since  $H^2(T_{X'}) = 0$  by [LP07], the cyclic quotient singularities can be smoothed independently. So there exists a  $\mathbb{Q}$ -Gorenstein smoothing such that the deformation of one  $\frac{1}{4}(1, 1)$  singularity is trivial and the remaining singularities are smoothed.

Then the resulting surface  $X$  is a  $\mathbb{Q}$ -Gorenstein degeneration of the Godeaux surface having a unique  $\frac{1}{4}(1, 1)$  singularity.  $\square$

**Proposition 2.9.** *Let  $Y$  be a Godeaux surface with  $H_1(Y) = \mathbb{Z}/2\mathbb{Z}$ , whose construction is described in [KLP10], Section 3, Example 1. Then there is a degeneration  $Y \rightsquigarrow X$ , where  $X$  has a unique singularity of type  $\frac{1}{4}(1, 1)$ , and  $K_X$  is ample. The surface  $X$  is obtained by smoothing all but one  $\frac{1}{4}(1, 1)$  singularity in the surface  $X'$  in the construction of Keum, Lee and Park.*

*Proof.* A surface  $X$  with  $H_1(X) = \mathbb{Z}/2\mathbb{Z}$  and such that its minimal resolution is an elliptic fibration with two multiple fibers of multiplicities 2 and 4 can be obtained by smoothing all but one  $\frac{1}{4}(1, 1)$  singularity in [KLP10], Example 3.1.

Keum, Lee and Park obtain a surface  $X'$  which has two cyclic quotient singularities of the type  $\frac{1}{27}(1, 8)$  and two cyclic quotient singularities of type  $\frac{1}{4}(1, 1)$  such that  $K_{X'}$  is ample. They prove that  $X'$  has a  $\mathbb{Q}$ -Gorenstein smoothing such that a general fiber of the family is a minimal, complex surface of general type with  $p_g = 0$ ,  $K^2 = 1$ , and  $H_1 = \mathbb{Z}/2\mathbb{Z}$ . See [KLP10], Section 3, Example 1 for details on the construction, as well as the outline of the proof.

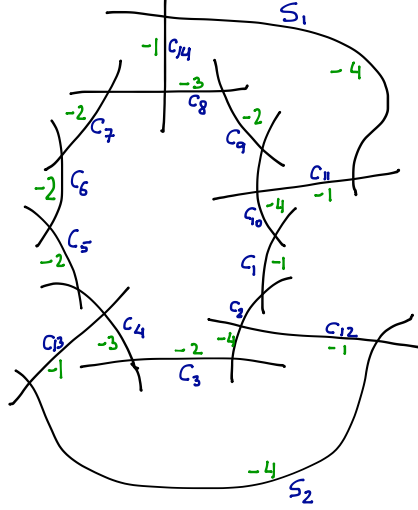
In particular, since  $H^2(T_{X'}) = 0$  [KLP10], the cyclic quotient singularities can be smoothed independently. So there exists a  $\mathbb{Q}$ -Gorenstein deformation such that the deformation of one  $\frac{1}{4}(1, 1)$  singularity is trivial and the remaining singularities are smoothed.  $\square$

**Remark 2.10.** Proposition 2.12 together with Proposition 4.3 imply that this degeneration  $X$  corresponds to the case when the minimal resolution  $\tilde{X}$  has two multiple fibers of multiplicity 2 and 4.

**Remark 2.11.** At the moment we are missing the construction of a surface  $X$  in case (c), i.e. such that the resolution  $\tilde{X}$  has two multiple fibers of multiplicities 2 and 6. Note that in this case the canonical line bundle  $K_{\tilde{X}}$  will be 2-divisible modulo torsion. We expect this degeneration to exist, and the construction of this degeneration would require an argument similar to [LP07] or [KLP10].

*Proof of the Theorem 1.2.* By Proposition 2.1 and Theorem 2.3 the minimal resolution  $\tilde{X}$  of  $X$  is a minimal elliptic surface with exactly two multiple fibers. In particular, there does not exist a degeneration  $Y \rightsquigarrow X$  if  $H_1(Y, \mathbb{Z}) = \mathbb{Z}/5\mathbb{Z}$ . Propositions 2.6, 2.7, 2.8, 2.9 provide the explicit constructions of degenerations in all



FIGURE 1. Special fiber on  $Z$ .

other cases. Thus there is at least one  $\frac{1}{4}(1, 1)$  boundary component for each other  $H_1(Y, \mathbb{Z})$ .  $\square$

**Proposition 2.12.** *Let  $Y \rightsquigarrow X$  be the degeneration of a smooth Godeaux surface  $Y$  with  $H_1(Y) = \mathbb{Z}/2\mathbb{Z}$  to a surface  $X$  with a unique Wahl singularity of type  $\frac{1}{4}(1, 1)$  obtained in the Proposition 2.9. Then the divisor  $K_X + \sigma$ , where  $0 \neq \sigma \in \text{Tors } H_2(X)$ , is not 2-divisible in  $H_2(X, \mathbb{Z})$ .*

*Proof.* Let  $\bar{Y}$  be the Enriques surface. Following [KLP10], we blow the surface  $\bar{Y}$  up at five points to obtain the surface  $\tilde{Z}$ , such that the rank of the class group  $\text{Cl } \tilde{Z}$  is equal to 15. Let  $\pi : \tilde{Z} \rightarrow X'$  be the map contracting the four chains of  $\mathbb{P}^1$ 's.

We would like show that  $K_X + \sigma$  is not 2-divisible in  $H_2(X, \mathbb{Z})$ .

Note that it is sufficient to show that  $K_{X'}$  is not 2-divisible in  $H_2(X', \mathbb{Z})/\text{Tors } X'$ . Indeed, we have the specialization map  $sp : H_2(X, \mathbb{Z}) \rightarrow H_2(X', \mathbb{Z})$ , such that  $sp(K_X) = K_{X'}$ .

Moreover, it suffices to show that  $K_{X'}$  is not 2-divisible in  $H_2(X', \mathbb{Z})/\text{Tors } X'$ , since there is a surjective map  $H_2(X', \mathbb{Z}) \rightarrow H_2(X', \mathbb{Z})/\text{Tors } X'$ , mapping  $(K_{X'} + \sigma) \mapsto [K_{X'}]$ . So we will consider everything up to a torsion element. Write  $L = \text{Cl } \tilde{Z}/\text{Tors } \tilde{Z}$ . On  $\tilde{Z}$  denote the curves in the fiber by  $C_1, \dots, C_{14}$ , and the two bisectors by  $S_1, S_2$  as shown in Figure 1.

Note that  $C_1, \dots, C_{14}$  are linearly independent in  $L$  since they form a fiber of the elliptic fibration, and denote by  $A = \langle C_1, \dots, C_{14}, S_1 \rangle$ . Then  $A$  has rank 15, so it has finite index in  $L$ . Using the intersection matrix for  $A$  we compute that the index  $|L/A| = 6$ .

Let  $M = \langle C_2, \dots, C_5, C_7, \dots, C_{10}, S_1, S_2 \rangle$  be the set of all  $\pi$ -exceptional divisors on  $\tilde{Z}$ , then we need to show that  $K_{X'}$  is not 2-divisible in  $\text{Cl}(X')/\pi_*(\text{Tors } \tilde{Z}) = L/M$

Let  $G = (1/2)F$ , where  $F$  is a general fiber of the fibration  $\tilde{Z} \rightarrow \mathbb{P}^1$ . Note that  $G$  is an element of  $L$  since  $\tilde{Z}$  has two multiple fibers of multiplicity 2 because it is a blowup of an Enriques surface [BHPV04], Chapter VIII, Lemma 17.1.

Consider  $N = A + \mathbb{Z} \cdot G$ , then we have  $A \subset N \subset L$ . Note that the containments are strict as we have  $G \notin A$ . To show this, note that  $G = (1/2)F \in F^\perp$  since  $F^2 = 0$ . But  $G \notin A \cap F^\perp = \langle C_1, \dots, C_{14} \rangle$ , since  $G = \frac{1}{2}(2C_1 + C_2 + \dots + C_{14})$ .

Thus we have  $|N/A| = 2$ , so  $|L/N| = 3$ . Moreover, a basis of  $N$  is given by  $\langle C_1, \dots, C_{13}, G, S_1 \rangle$ .

Unfortunately  $M \not\subset N$ , but since  $|L/N| = 3$ , we have  $N \otimes \mathbb{Z}/2\mathbb{Z} \simeq L \otimes \mathbb{Z}/2\mathbb{Z}$ . So it is enough to show that  $K_{X'}$  is nonzero in  $(N \otimes \mathbb{Z}/2\mathbb{Z})/(M \otimes \mathbb{Z}/2\mathbb{Z})$  to conclude that it is not 2-divisible in  $L/M$ .

On  $\tilde{Z}$  we have  $S_i \cdot F = 2$ ,  $i = 1, 2$ , and  $C_i \in F^\perp$ . Note that  $S_1 - S_2 \in F^\perp$ , so we can write  $S_1 - S_2 = \sum_{i=1}^{14} x_i C_i$  for some  $x_i \in \mathbb{Q}$ . Using the intersection matrix  $\langle C_i \cdot C_j \rangle$  we can compute the vector

$$\mathbf{x} = (x_1, \dots, x_{14}) = \frac{-1}{3}(14, 5, 4, 3, 5, 7, 9, 11, 10, 9, 12, 2, 14, 0) + \lambda/6(2, 1, \dots, 1)$$

for some  $\lambda \in \mathbb{Z}$ , since we can write the general fiber  $F = 2G = \sum_{i=1}^{14} m_i C_i$ , where  $m_1 = 2$  and  $m_i = 1$  for all  $i \neq 2$ .

Thus modulo 2 we have  $S_1 - S_2 = C_2 + C_4 + C_5 + C_6 + C_7 + C_8 + C_{10} + \mu G \in N$ , for some  $\mu \in \mathbb{Z}$ .

So

$$\begin{aligned} \frac{N \otimes \mathbb{Z}/2\mathbb{Z}}{M \otimes \mathbb{Z}/2\mathbb{Z}} &= \frac{\langle C_1, \dots, C_{13}, G, S_1 \rangle \otimes \mathbb{Z}/2\mathbb{Z}}{\langle C_2, \dots, C_5, C_7, \dots, C_{10}, S_1, S_1 - S_2 \rangle \otimes \mathbb{Z}/2\mathbb{Z}} = \\ &= \frac{\langle C_1, C_6, C_{11}, C_{12}, C_{13}, G \rangle}{\langle C_6 + \mu G \rangle} \otimes \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

We have  $K_{\tilde{Z}} = p^* K_{\tilde{Y}} + (C_1 + C_{11} + C_{12} + C_{13} + C_{14}) = C_1 + C_{11} + C_{12} + C_{13} + C_{14}$  in  $L$ . Then  $K_{X'} = \pi_* K_{\tilde{Z}} \in \text{Cl}(X')/\pi_* \text{Tors } \tilde{Z} = L/M$ , so  $K_{X'} = C_1 + C_{11} + C_{12} + C_{13} + C_{14}$ .

Finally in  $(N \otimes \mathbb{Z}/2\mathbb{Z})/(M \otimes \mathbb{Z}/2\mathbb{Z}) \simeq (\mathbb{Z}/2\mathbb{Z})^6/\langle 0, 1, 0, 0, 0, \mu \rangle$  we have  $K_{X'} \equiv 2G - C_1 - C_6$ , so  $K_{X'} \equiv C_1 + C_6$ . Thus  $K_{X'}$  is nonzero in  $(N \otimes \mathbb{Z}/2\mathbb{Z})/(M \otimes \mathbb{Z}/2\mathbb{Z})$ , so it is not divisible in  $H_2(X', \mathbb{Z})$ .  $\square$

### 3. EXCEPTIONAL VECTOR BUNDLES

In this section we study  $K_Y$  stable exceptional vector bundles of rank 2 on smooth Godeaux surfaces  $Y$ .

**Definition 3.1.** An *exceptional vector bundle*  $E$  on a surface  $Y$  is a holomorphic vector bundle such that  $\text{Hom}(E, E) = \mathbb{C}$  and  $\text{Ext}^1(E, E) = \text{Ext}^2(E, E) = 0$ .

**Definition 3.2.** The *slope* of a vector bundle  $E$  of rank  $r$  on a surface  $Y$  with respect to an ample line bundle  $H$  is

$$(6) \quad \mu(E) = \frac{c_1(E) \cdot H}{r}.$$

**Definition 3.3.** A vector bundle  $E$  on a surface  $Y$  is called *stable with respect to an ample divisor  $H$*  if for every vector bundle  $F \hookrightarrow E$ , such that  $0 < \text{rank}(F) < \text{rank}(E)$  we have  $\mu(F) < \mu(E)$ .

**Definition 3.4.** Let  $E$  and  $F$  be two vector bundles on a smooth projective surface  $Y$ . Define

$$\chi(E, F) = \sum_{i=0}^2 (-1)^i \operatorname{Ext}^i(E, F).$$

We will use the following two facts.

**Lemma 3.5.** *Let  $E$  be a vector bundle on a smooth projective surface  $Y$ . Then  $E$  is exceptional if and only if the dual vector bundle  $E^\vee$  is exceptional. Moreover, for any line bundle  $L$  on  $Y$  we have  $E$  is exceptional if and only if  $E \otimes L$  is exceptional.*

*Proof.* We have  $\operatorname{Ext}^i(E, E) = H^i(\operatorname{Hom}(E, E))$ , as well as  $\operatorname{Hom}(E \otimes L, E \otimes L) = \operatorname{Hom}(E, E)$ , also  $\operatorname{Hom}(E^\vee, E^\vee) = \operatorname{Hom}(E, E)$ .  $\square$

**Lemma 3.6.** *Let  $Y$  be a smooth surface with  $\chi(\mathcal{O}_Y) = 1$ . Let  $E$  be an exceptional vector bundle of rank  $n$  on  $Y$ . Then*

$$(7) \quad c_2(E) = \frac{n-1}{2n} (c_1(E)^2 + n + 1)$$

*Proof.* On one hand we have  $\chi(E, E) = \operatorname{Hom}(E, E) - \operatorname{Ext}^1(E, E) + \operatorname{Ext}^2(E, E) = 1$ , since  $E$  is exceptional. Also by the Hirzebruch–Riemann–Roch formula we have

$$1 = \chi(\mathcal{E}nd E) = n^2 \chi(\mathcal{O}_Y) + (n-1)c_1(E)^2 - 2nc_2(E).$$

We can get the formula (7) by solving this equation for  $c_2(E)$ .  $\square$

In [Hac13a] Hacking describes a way to construct exceptional vector bundles of rank  $n$  on a smooth surface  $Y$  such that  $H^1(\mathcal{O}_Y) = H^2(\mathcal{O}_Y) = 0$  using degenerations  $Y \rightsquigarrow X$  of  $Y$  to a surface  $X$  with a unique singularity of Wahl type  $\frac{1}{n^2}(1, na-1)$ . We quote the following theorem of Hacking.

**Theorem 3.7** ([Hac13a], Thm. 1.1). *Let  $\mathcal{X}/(0 \in S)$  be a one parameter  $\mathbb{Q}$ -Gorenstein smoothing of a normal projective surface  $X$  with a unique singularity  $(P \in X)$  of Wahl type  $\frac{1}{n^2}(1, na-1)$ . Let  $Y$  denote a general fiber of  $\mathcal{X}/(0 \in S)$ . Assume that  $H^1(\mathcal{O}_Y) = H^2(\mathcal{O}_Y) = 0$  and the map  $H_1(Y, \mathbb{Z}) \rightarrow H_1(X, \mathbb{Z})$  is injective. Then after a finite base change  $S' \rightarrow S$  there is a rank  $n$  reflexive sheaf  $\mathcal{E}$  on  $\mathcal{X}'$  such that  $E := \mathcal{E}|_Y$  is an exceptional bundle on  $Y$ . The topological invariants of  $E$  are given by:  $\operatorname{rank}(E) = n$ ,  $c_1(E) \cdot K_Y = \pm a \pmod{n}$ , and  $c_2(E) = \frac{n-1}{2n} (c_1(E)^2 + n + 1)$ .*

*If  $\mathcal{H}$  is an ample line bundle on  $\mathcal{X}$  over  $S$ , then  $E$  is slope stable with respect to  $\mathcal{H}|_Y$ .*

For a  $K_Y$ -stable exceptional vector bundle  $E$  as produced by Theorem 3.7, we will be most interested in its slope vector, which is the numerical invariant  $c_1(E)/\operatorname{rank}(E) \in H^2(Y, \mathbb{Q})$ . Furthermore, we will say that two slope vectors are equivalent “ $\sim$ ” if they differ a combination of translation by  $H^2(Y, \mathbb{Z})$ , multiplication by  $\pm 1$ , and the action of the monodromy group  $\Gamma \subset \operatorname{Aut}(H^2(Y, \mathbb{Z}))$ . (The first two correspond to the vector bundle operations  $E \rightsquigarrow E \otimes L$  for a line bundle  $L$ , and  $E \rightsquigarrow E^*$ .)

In this section we study exceptional vector bundles  $E$  of rank 2 on a Godeaux surface  $Y$  such that  $E$  is stable with respect to  $K_Y$  up to the equivalence relation defined in Remark 3.8. Note that these exceptional vector bundles correspond to the degenerations of smooth Godeaux surfaces to surfaces with a unique singularity of type  $\frac{1}{4}(1, 1)$  in the construction of Hacking.

**Remark 3.8.** Since we are only dealing with vector bundles of the same rank, we can redefine the equivalence relation “ $\sim$ ” from Theorem 3.7 as follows. Two exceptional vector bundles  $E_1$  and  $E_2$  of the same rank are equivalent if  $c_1(E_1)$  can be obtained from  $c_1(E_2)$  by translation by  $\text{rank}(E)H^2(Y, \mathbb{Z})$ , multiplication by  $\pm 1$  and the action of the monodromy group. So we will be thinking of the equivalence relation  $\sim$  as a relation on  $c_1(E) \in H^2(Y, \mathbb{Z})/\text{Tors } Y$ .

Then we can compute orbits of  $c_1(E)$  under the equivalence relation  $\sim$ .

The lattice  $B := H^2(Y, \mathbb{Z})/\text{Tors } Y$  of a Godeaux surface  $Y$  is isomorphic to  $\mathbb{Z}K_Y \oplus (-E_8)$ . We can translate  $E$  by a line bundle  $\mathcal{O}(M)$  for some  $M \in H^2(Y, \mathbb{Z})$  so that  $c_1(E)$  can be replaced by  $c_1(E \otimes \mathcal{O}(M)) = c_1(E) + 2c_1(\mathcal{O}(M))$ . Thus  $c_1(E) \in H^2(Y, \mathbb{Z}) \bmod \text{Tors } Y$  can be considered as  $c_1(E) \in B/2B \simeq (\mathbb{Z}/2\mathbb{Z})^9$ .

The monodromy group here is

$$\Gamma \subseteq \text{Aut}(B, K_Y) = \{g \in \text{Aut } B \mid g(K_Y) = K_Y\}.$$

We have  $\text{Aut}(B, K_Y) = \text{Aut}(-E_8)$ , and the automorphism group  $\text{Aut}(-E_8)$  is equal to the Weyl group  $W(E_8)$ . We expect the monodromy group to be equal to  $W(E_8)$ . Under this assumption the following lemma holds.

**Lemma 3.9.** *Suppose that the monodromy group is equal to  $W(E_8)$ . Up to the equivalence relation generated by translation by an element of  $H^2(Y, \mathbb{Z})$ , and the monodromy group, there are two equivalence classes of  $c_1(E) \in B$  with  $c_1(E)^2 \equiv 1 \bmod 4$ , one is given by  $c_1(E) \sim K_Y$  modulo torsion.*

*Proof.* Note that  $(H^2(Y, \mathbb{Z})/\text{Tors } Y, \cup) = (H^2(\text{Bl}^8 \mathbb{P}^2), \cup)$ , so instead of computing the orbits of the action of the monodromy group on  $(H^2(Y, \mathbb{Z})/\text{Tors } Y, \cup)$ , we can find them for  $(H^2(\text{Bl}^8 \mathbb{P}^2), \cup)$ . The lattice  $(H^2(\text{Bl}^8 \mathbb{P}^2), \cup)$  has basis  $\pi^*(H)$ ,  $E_1, \dots, E_8$ , where  $\pi : \text{Bl}^8 \mathbb{P}^2 \rightarrow \mathbb{P}^2$ ,  $H \subset \mathbb{P}^2$  is a hyperplane, and  $E_1, \dots, E_8$  are exceptional divisors on  $\text{Bl}^8 \mathbb{P}^2$  satisfying  $E_i^2 = -1$  and  $E_i \cdot E_j = 0$  for  $i \neq j$ . Then for the  $\text{Aut}(E_8)$  group action on the lattice  $H^2(\text{Bl}^8 \mathbb{P}^2)/2H^2(\text{Bl}^8 \mathbb{P}^2)$  there are only 2 orbits (given by  $\pi^*(H)$  and  $-3\pi^*(H) + E_1 + \dots + E_8 = K_{\text{Bl}^8 \mathbb{P}^2}$ ). Thus there are only two corresponding orbits in  $(H^2(Y, \mathbb{Z}), \cup)$ .  $\square$

We use the following construction of rank 2 vector bundles.

**Theorem 3.10** ([HL10], Theorem 5.1.1). *Let  $Z \subset X$  be a local complete intersection of codimension two, and let  $M$  and  $L$  be line bundles on  $X$ . Then there exists an extension*

$$0 \rightarrow L \rightarrow E \rightarrow M \otimes \mathcal{I}_Z \rightarrow 0$$

*such that  $E$  is locally free if and only if the pair  $(L^\vee \otimes M \otimes K_X, Z)$  has the Cayley–Bacharach property:*

*(CB) If  $Z' \subset Z$  is a subscheme with  $\text{length}(Z') = \text{length}(Z) - 1$  and  $s \in H^0(X, L^\vee \otimes M \otimes K_X)$  is a section with  $s|_{Z'} = 0$ , then  $s|_Z = 0$ .*

*Moreover,  $c_1(E) = c_1(L) + c_1(M)$ , and  $c_2(E) = c_1(L) \cdot c_1(M) + \text{length}(Z)$ .*

After a twist by a line bundle ( $E \mapsto M \otimes E$ ) there exists a section  $s \in \Gamma(Y, E)$  with isolated zeroes, so any rank two vector bundle on a surface  $Y$  can be described as an extension

$$(8) \quad 0 \rightarrow \mathcal{O}_Y \rightarrow E \rightarrow L \otimes \mathcal{I}_Z \rightarrow 0,$$

where  $L = c_1(E)$  is a line bundle on the surface  $Y$ ,  $Z$  is a zero dimensional subscheme of  $Y$  with  $\text{length}(Z) = c_2(E)$ , and the divisor  $K_Y + L$  satisfies the Cayley-Bacharach property with respect to  $Z$ .

The following technical lemma gives us a method to check if the vector bundle constructed using the exact sequence (8) is exceptional.

**Lemma 3.11.** *Let  $Y$  be a Godeaux surface with  $K_Y$  ample. Suppose that a vector bundle  $E$  of rank 2 on  $Y$  and such that  $c_2(E) = \frac{1}{4}(c_1(E)^2 + 3)$  is given by an exact sequence*

$$(9) \quad 0 \rightarrow \mathcal{O}_Y \rightarrow E \rightarrow L \otimes \mathcal{I}_Z \rightarrow 0.$$

*Assume that  $L \cdot K_Y > 0$ . Then  $E$  is exceptional if  $Z \neq \emptyset$ ,  $L \neq K_Y$ ,  $H^0(L \otimes \mathcal{I}_Z) = 0$ , and  $H^0(\mathcal{O}_Y(K_Y) \otimes L \otimes \mathcal{I}_Z^2) = 0$ .*

*Proof.* By the Riemann–Roch theorem and our assumption that  $c_2(E) = \frac{1}{4}(c_1(E)^2 + 3)$  we have  $\chi(\mathcal{E}nd E) = 1$ , see Theorem 3.6. Thus it is enough to prove that  $\text{Hom}(E, E) = \mathbb{C}$  and  $\text{Ext}^2(E, E) = 0$ . By the decomposition  $H^2(Y, \mathbb{Z})/\text{Tors} = \langle K_Y \rangle \oplus \langle K_Y^\perp \rangle$  we can write  $L = mK_Y + A$ , where  $A \in K_Y^\perp$ . Here  $m = L \cdot K_Y$  and we assume  $m > 0$ .

First let us examine the condition  $\text{Hom}(E, E) = \mathbb{C}$ . By applying  $\text{Hom}(-, E)$  to (9) we obtain the exact sequence

$$0 \rightarrow H^0(E \otimes L^\vee) \rightarrow \text{Hom}(E, E) \rightarrow H^0(E) \rightarrow \text{Ext}^1(L \otimes \mathcal{I}_Z, E) \rightarrow \dots$$

By tensoring (9) with  $L^\vee$  we get

$$0 \rightarrow L^\vee \rightarrow E \otimes L^\vee \rightarrow \mathcal{I}_Z \rightarrow 0.$$

Note that  $H^0(L^\vee) = 0$  since  $K_Y$  is ample, and  $L \cdot K_Y > 0$ . Also  $H^0(\mathcal{I}_Z) = 0$  because  $Z \neq \emptyset$  and  $H^0(\mathcal{O}_Y) = \mathbb{C}$ . Therefore  $H^0(E \otimes L^\vee) = 0$ . By (9)  $H^0(E)/H^0(\mathcal{O}_Y) \simeq H^0(L \otimes \mathcal{I}_Z)$ , using  $H^1(\mathcal{O}_Y) = 0$ . So the condition  $\text{Hom}(E, E) = \mathbb{C}$  is implied by our assumption

$$(10) \quad H^0(L \otimes \mathcal{I}_Z) = 0.$$

Now consider the condition  $\text{Ext}^2(E, E) = 0$ . Write  $\omega_Y = \mathcal{O}_Y(K_Y)$ . Note that  $\text{Ext}^2(E, E) = \text{Hom}(E, E \otimes \omega_Y)^*$  by Serre duality, so it suffices to prove that  $\text{Hom}(E, E \otimes \omega_Y)$  is trivial.

Tensoring the exact sequence (9) with  $\omega_Y$ , we obtain the following exact sequence

$$(11) \quad 0 \rightarrow \omega_Y \rightarrow E \otimes \omega_Y \rightarrow \omega_Y \otimes L \otimes \mathcal{I}_Z \rightarrow 0.$$

Now we can apply  $\mathcal{H}om(E, -)$  to the exact sequence (11) using the fact that  $\mathcal{H}om(E, -) = E^\vee \otimes - = E \otimes L^\vee \otimes -$ . Here  $E^\vee \simeq E \otimes L^\vee$ , since  $\text{rank } E = 2$ , and  $L = \det E = \wedge^2 E$ , so the perfect pairing  $E \otimes E \rightarrow \wedge^2 E = L$  induces an isomorphism  $E \simeq E^\vee \otimes L$ . The result will be an exact sequence again since  $E$  is locally free:

$$0 \rightarrow E \otimes L^\vee \otimes \omega_Y \rightarrow \mathcal{H}om(E, E \otimes \omega_Y) \rightarrow E \otimes \omega_Y \otimes \mathcal{I}_Z \rightarrow 0.$$

By tensoring the exact sequence (9) with  $L^\vee \otimes \omega_Y$  we obtain that

$$H^0(E \otimes L^\vee \otimes \omega_Y) = 0,$$

because  $H^0(\mathcal{O}_Y(-(m-1)K_Y - A)) = 0$  for all  $m > 1$ , and  $H^0(\mathcal{O}_Y(K_Y)) = 0$ . In the case  $m = 1$  we have  $H^0(\mathcal{O}_Y(-A)) = 0$  unless  $A = 0$ , because  $A \cdot K_Y = 0$ , and  $K_Y$  is ample. By Lemma 3.14 below,  $E$  is not exceptional in the remaining case  $A = 0$ , i.e.  $L = K_Y$ .

So it suffices to show that  $H^0(E \otimes \omega_Y \otimes \mathcal{I}_Z) = 0$ . We will show that

$$H^0(E \otimes \omega_Y \otimes \mathcal{I}_Z) \simeq H^0(\omega_Y \otimes L \otimes \mathcal{I}_Z^2).$$

By tensoring  $(E \otimes \omega_Y)$  with the exact sequence  $0 \rightarrow \mathcal{I}_Z \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_Z \rightarrow 0$  we obtain the exact sequence

$$0 \rightarrow E \otimes \omega_Y \otimes \mathcal{I}_Z \rightarrow E \otimes \omega_Y \rightarrow E \otimes \omega_Y|_Z \rightarrow 0$$

and the associated long exact sequence of cohomology

$$0 \rightarrow H^0(E \otimes \omega_Y \otimes \mathcal{I}_Z) \rightarrow H^0(E \otimes \omega_Y) \rightarrow H^0((E \otimes \omega_Y)|_Z) \rightarrow \dots$$

Now using (11) we get  $H^0(E \otimes \omega_Y) \simeq H^0(\omega_Y \otimes L \otimes \mathcal{I}_Z)$  since  $H^0(\omega_Y) = H^1(\omega_Y) = 0$ .

Also we can restrict the sequence (11) to  $Z$  to obtain an exact sequence

$$\omega_Y|_Z \rightarrow E \otimes \omega_Y|_Z \rightarrow \omega_Y \otimes L \otimes \mathcal{I}_Z|_Z \rightarrow 0.$$

The map  $s : \omega_Y \rightarrow E \otimes \omega_Y$  is defined by a section of  $E$  which vanishes on  $Z$ , so the map  $\omega_Y|_Z \rightarrow E \otimes \omega_Y|_Z$  is the zero map, thus  $H^0(E \otimes \omega_Y|_Z) \simeq H^0(\omega_Y \otimes L \otimes \mathcal{I}_Z|_Z)$ .

By tensoring the exact sequence  $0 \rightarrow \mathcal{I}_Z \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_Z \rightarrow 0$  with  $\mathcal{I}_Z$ , we obtain  $\mathcal{I}_Z \otimes \mathcal{I}_Z \rightarrow \mathcal{I}_Z \rightarrow \mathcal{I}_Z|_Z \rightarrow 0$ , or

$$(12) \quad 0 \rightarrow \mathcal{I}_Z^2 \rightarrow \mathcal{I}_Z \rightarrow \mathcal{I}_Z|_Z \rightarrow 0$$

We have the following commutative diagram with exact rows

$$\begin{array}{ccccc} 0 \rightarrow H^0(E \otimes \omega_Y \otimes \mathcal{I}_Z) & \longrightarrow & H^0(E \otimes \omega_Y) & \longrightarrow & H^0((E \otimes \omega_Y)|_Z) \\ & & \downarrow a & & \downarrow c \\ 0 \rightarrow H^0(L \otimes \omega_Y \otimes \mathcal{I}_Z^2) & \longrightarrow & H^0(L \otimes \omega_Y \otimes \mathcal{I}_Z) & \longrightarrow & H^0((L \otimes \omega_Y \otimes \mathcal{I}_Z)|_Z), \end{array}$$

where the bottom row is obtained by tensoring (12) with  $(L \otimes \omega_Y)$ , and then taking global sections. Since the maps  $b$  and  $c$  are isomorphisms, we conclude that  $a$  is an isomorphism as well.

Thus  $\text{Ext}^2(E, E) = 0$  if

$$(13) \quad H^0(\omega_Y \otimes L \otimes \mathcal{I}_Z^2) = 0.$$

□

Our analysis of exceptional vector bundles on a Godeaux surface will be divided into two cases corresponding to the two orbits from Lemma 3.9. We first handle the case when  $c_1(E) \sim K_Y \pmod{\text{Tors } Y}$  by the following theorem. We state more precise version of the Theorem 1.3 here.

**Theorem 3.12.** *Let  $Y$  be a Godeaux surface with  $K_Y$  ample,  $\sigma \in \text{Tors } Y$  and  $P$  a base point of  $|2K_Y + \sigma|$ . There is a unique vector bundle  $E$  of rank 2 on  $Y$  given by an extension*

$$(14) \quad 0 \rightarrow \mathcal{O}_Y \rightarrow E \rightarrow \mathcal{O}_Y(K_Y + \sigma) \otimes \mathcal{I}_P \rightarrow 0.$$

Then

- $E$  is stable with respect to  $K_Y$ ,  $c_1(E) = K_Y + \sigma$ , and  $c_2(E) = 1$ .
- Conversely, every  $K_Y$ -stable rank 2 vector bundle with  $c_1(E) = K_Y$  modulo torsion and  $c_2(E) = 1$  is equivalent to one of this form.
- $E$  is exceptional if  $\sigma \in \text{Tors } Y \setminus 2\text{Tors } Y$  (in particular, we require  $|\text{Tors } Y|$  is even), and  $P$  is a simple base point of  $|2K_Y + \sigma|$ .
- If  $E$  is exceptional, then  $\sigma \in \text{Tors } Y \setminus 2\text{Tors } Y$

The proof breaks into three lemmas.

**Lemma 3.13.** *We can define a vector bundle  $E$  using the exact sequence (14), and such a vector bundle is stable with respect to  $K_Y$ . Moreover,  $E$  is uniquely determined by  $\sigma$  and  $P$ . Conversely, if  $E$  is exceptional vector bundle with  $c_1(E) = K_Y \bmod \text{Tors } Y$  and  $c_2(E) = 1$ , stable with respect to  $K_Y$ , then up to  $E \rightsquigarrow E(\tau)$ ,  $\tau \in \text{Tors } Y$ , it is given by an exact sequence (14).*

*Proof.* The (CB) condition in this case is satisfied for  $P$  being a base point of  $|K_Y + L| = |2K_Y + \sigma|$ . So we can define a vector bundle  $E$  using (14). We need to show that such a vector bundle  $E$  is stable with respect to  $K_Y$ . Suppose that  $E$  is not stable with respect to  $K_Y$ . Then since  $\mu(E) = \frac{1}{2}c_1(E) \cdot K_Y = \frac{1}{2}$ , there is an effective divisor  $D$  such that  $D \cdot K_Y > 0$  and a nonzero map  $\mathcal{O}_Y(D) \hookrightarrow E$ . Write  $D = nK_Y + \beta$ , where  $n = D \cdot K_Y > 0$  and  $\beta \in K_Y^\perp$ . Consider the exact sequence

$$0 \rightarrow \mathcal{O}_Y(-D) \rightarrow E(-D) \rightarrow \mathcal{O}_Y(K_Y + \sigma) \otimes \mathcal{O}_Y(-D) \otimes \mathcal{I}_P \rightarrow 0,$$

obtained by tensoring (14) by  $\mathcal{O}_Y(-D)$ .

We have  $H^0(\mathcal{O}_Y(-D)) = 0$  because  $D \cdot K_Y > 0$ , and

$$H^0(E(-D)) = \text{Hom}(\mathcal{O}_Y(D), E) \neq 0$$

by assumption, so

$$H^0(\mathcal{O}_Y(K_Y + \sigma - nK_Y - \beta) \otimes \mathcal{I}_P) \neq 0.$$

This is impossible for  $n > 1$ . If  $n = 1$  then  $H^0(\mathcal{O}_Y(\sigma - \beta)) = 0$  unless we have  $\sigma - \beta = 0$ . But then  $H^0(\mathcal{O}_Y \otimes \mathcal{I}_P) = H^0(\mathcal{I}_P) = 0$ . So such a divisor  $D$  does not exist and  $E$  is stable with respect to  $K_Y$ .

Conversely, if  $E$  is an exceptional vector bundle of rank 2 with  $c_1(E) = K + \sigma$ , then according to the formula (7), the second Chern class  $c_2(E)$  is fixed and is equal to 1.

We need to check that  $E$  can be defined by the exact sequence (14) for some  $\sigma$  and  $P$ . First, let us show that there exists a section  $s \in H^0(E)$ , or equivalently, that  $h^0(E) > 0$ . By the Riemann–Roch formula  $\chi(E) = 1$ . Also by Serre Duality  $h^2(E) = h^0(E^\vee \otimes K) = h^0(E(-\sigma))$  since we have  $E \simeq E^\vee \otimes \det E = E^\vee \otimes \mathcal{O}_Y(K_Y + \sigma)$ .

So  $0 < \chi(E) = h^0(E) - h^1(E) + h^0(E(-\sigma)) \leq h^0(E) + h^0(E(-\sigma))$ . Thus  $h^0(E) > 0$ , or  $h^0(E(-\sigma)) > 0$ , so, possibly after a twist  $E \rightsquigarrow E(-\sigma)$ , there is a nonzero global section of  $E$ .

Now we need to check that a nonzero global section of  $E$  has isolated zeroes, assuming that  $E$  is  $K_Y$ -stable. Suppose that it does not, so there is a map  $\mathcal{O}(D) \rightarrow E$  for some effective divisor  $D$ . But then  $\mu(\mathcal{O}(D)) = D \cdot K \geq 1$  and  $\mu(E) = \frac{1}{2}K^2 = \frac{1}{2}$ , which contradicts stability of  $E$ .

Finally, we have

$$0 \rightarrow \mathcal{O}_Y \rightarrow E \rightarrow L \otimes \mathcal{I}_Z \rightarrow 0,$$

where  $c_1(E) = L$ ,  $c_2(E) = \text{length}(Z)$ . Then  $L = K_Y + \sigma$ , for some  $\sigma \in \text{Tors } Y$ , and  $Z = P$  is a reduced point. Now by Theorem 3.10,  $P$  is a base point of  $|2K_Y + \sigma|$ .

Now let us show that such  $E$  is unique. It suffices to show that  $\text{Ext}^1(\mathcal{O}_Y(K_Y + \sigma) \otimes \mathcal{I}_P, \mathcal{O}_Y) \simeq \mathbb{C}$ . By local-to-global spectral sequence for  $\text{Ext}$  we have an exact sequence.

$$0 \rightarrow H^1(\mathcal{H}om) \rightarrow \text{Ext}^1 \rightarrow H^0(\mathcal{E}xt^1) \rightarrow H^2(\mathcal{H}om)$$

Now  $\text{Hom}(\mathcal{O}_Y(K_Y + \sigma) \otimes \mathcal{I}_P, \mathcal{O}_Y) = \mathcal{O}_Y(-K_Y - \sigma)$ , and  $H^1(\mathcal{O}_Y(-K_Y - \sigma)) = 0$  by Kodaira vanishing. Also  $\mathcal{E}xt^1(\mathcal{O}_Y(K_Y + \sigma) \otimes \mathcal{I}_P, \mathcal{O}_Y) \simeq \mathcal{E}xt^1(\mathcal{I}_P, \mathcal{O}_Y) \simeq k(P)$ , so that  $H^0(\mathcal{E}xt^1(\mathcal{O}_Y(K_Y + \sigma) \otimes \mathcal{I}_P, \mathcal{O}_Y)) \simeq \mathbb{C}$ .

So we have an exact sequence

$$0 \rightarrow \text{Ext}^1 \xrightarrow{\alpha} H^0(\mathcal{E}xt^1) \rightarrow H^2(\mathcal{H}om)$$

Since the Cayley–Bacharach condition is satisfied, there exists an extension which is a vector bundle. So  $\alpha \neq 0$ , thus  $\text{Ext}^1(\mathcal{O}_Y(K_Y + \sigma) \otimes \mathcal{I}_P, \mathcal{O}_Y) \simeq \mathbb{C}$ .  $\square$

**Lemma 3.14.** *If  $c_1(E) = K_Y + \sigma$ , where  $\sigma \in 2\text{Tors } Y$ , then the vector bundle  $E$  defined by the exact sequence (14) is not exceptional.*

*Proof.* First suppose that  $c_1(E) = K_Y$ . We have inclusions  $\mathcal{O}_Y(K_Y) \otimes \mathcal{I}_P \subset \mathcal{O}_Y(K_Y) \subset E \otimes \mathcal{O}_Y(K_Y)$  and a surjection  $E \rightarrow \mathcal{O}_Y(K_Y) \otimes \mathcal{I}_P$  coming from the exact sequence (14). Thus there exists a nonzero map  $f : E \rightarrow E \otimes \mathcal{O}_Y(K_Y)$ , defined by the composition of surjection followed by inclusion. But  $\text{Ext}^2(E, E) = H^2(\mathcal{H}om(E, E)) = H^0(\mathcal{H}om(E, E)^\vee \otimes \omega_Y)^* = \text{Hom}(E, E \otimes \omega_Y)^*$  by Serre duality. Thus since  $f \in \text{Hom}(E, E \otimes \omega_Y)$ , we conclude that  $\text{Ext}^2(E, E) \neq 0$ , so  $E$  is not exceptional.

If  $\sigma = 2\tau \in \text{Tors } Y$ , then  $E \otimes \mathcal{O}(-\tau)$  has  $c_1 = K_Y$ , thus  $E \otimes \mathcal{O}_Y(-\tau)$  is not exceptional, so  $E$  is not exceptional by Theorem 3.5.  $\square$

**Lemma 3.15.** *Let  $Y$  be a Godeaux surface,  $\sigma \in \text{Tors } Y/2\text{Tors } Y$ ,  $P \in |2K_Y + \sigma|$ , and  $E$  the associated vector bundle defined by (14). Then  $E$  is exceptional if  $P$  is a smooth point of the base locus of  $|2K_Y + \sigma|$ .*

*Proof.* Here we have  $L = \mathcal{O}_Y(K_Y + \sigma)$ . By Lemma 3.11 we only need to check that conditions (10) and (13) are satisfied.

Let  $P$  be a point in the base locus. First, we show that the condition (10) is satisfied for all Godeaux surfaces  $Y$ . By [Rei78], Lemma 0.3, we have  $H^0(K_Y + \sigma) \simeq \mathbb{C}$ , therefore  $|K_Y + \sigma| \neq \emptyset$ . Let  $C \in |K_Y + \sigma|$ , then  $C$  is an irreducible curve since  $K_Y \cdot C = K_Y^2 = 1$ . Suppose that  $P \in C$ , then  $(2K_Y + \sigma)|_C = K_C$  by the adjunction formula. More precisely,  $\omega_C = \omega_Y^{\otimes 2}(\sigma)|_C$ , where  $\omega_C$  is the dualizing sheaf of  $C$ . We have an exact sequence

$$0 \rightarrow \omega_Y \rightarrow \omega_Y^{\otimes 2}(\sigma) \rightarrow \omega_C \rightarrow 0,$$

so  $H^0(\omega_Y^{\otimes 2}(\sigma)) \rightarrow H^0(\omega_C)$  because  $H^1(\omega_Y) = H^1(\mathcal{O}_Y)^* = 0$ . It follows that  $P$  is a base point of  $\omega_C$ .

By the adjunction formula

$$2p_a(C) - 2 = \deg \omega_C = (K_Y + C) \cdot C = (2K_Y + \sigma) \cdot (K_Y + \sigma) = 2,$$

so  $p_a(C) = 2$ .

But by [Har86], Theorem 1.6 if  $C$  is a projective, irreducible Gorenstein curve with  $p_a(C) > 0$ , then  $\omega_C$  is base point free. Thus  $P \notin C$ , so  $H^0(\mathcal{O}_Y(K_Y + \sigma) \otimes \mathcal{I}_P) = 0$  and  $\text{Hom}(E, E) = \mathbb{C}$ .

So we just need to check the condition (13). By [Rei78], Lemma 0.3, we have  $\dim H^0(2K_Y + \sigma) = 2$ , so  $|2K_Y + \sigma| \simeq \mathbb{P}^1$ . Then  $P$  is a simple base point of  $|2K_Y + \sigma|$  if and only if there exist  $C_1$  and  $C_2 \in |2K_Y + \sigma|$  given locally by  $(x = 0)$  and  $(y = 0)$ , where  $x$  and  $y$  are some local coordinates at  $P$ , and  $P = C_1 \cap C_2$ , so



that the general  $C \in |2K_Y + \sigma|$  is given locally by  $(\lambda x + \mu y = 0)$  for some  $\lambda, \mu \in \mathbb{C}$ . This is equivalent to  $H^0(\mathcal{O}_Y(2K_Y + \sigma) \otimes \mathcal{I}_P^2) = 0$ .  $\square$

This completes the proof of the Theorem 3.12.

In the proposition below we provide an explicit construction of a Godeaux surface satisfying all assumptions of Theorem 3.12.

**Proposition 3.16.** *There exists a Godeaux surface  $Y$  with  $H_1(Y, \mathbb{Z}) = \mathbb{Z}/4\mathbb{Z}$ , such that  $|2K_Y + \sigma|$  has simple base points, where  $\sigma \in \text{Tors } Y$  is a torsion element of order 4.*

*Proof.* We use the notations of [Rei78] to describe such a surface.

In the weighted projective space  $\mathbb{P}(1, 1, 1, 2, 2)$  with coordinates  $x_1, x_2, x_3, y_1, y_3$  consider the surface  $F_{4,4} = \{q_0 = q_2 = 0\} \subset \mathbb{P}(1, 1, 1, 2, 2)$  defined by the complete intersection of two quartics

$$\begin{aligned} q_0 &= x_1^4 + x_2^4 + x_3^4 + y_1 y_3, \\ q_2 &= x_1^3 x_3 + x_1 x_3^3 + x_1 x_2 y_3 + y_1^2 + y_3^2. \end{aligned}$$

Consider the action of  $\mathbb{Z}/4\mathbb{Z}$  on  $F_{4,4}$  induced from the action of  $\mathbb{Z}/4\mathbb{Z}$  on the weighted projective space  $\mathbb{P}(1, 1, 1, 2, 2)$  defined by  $x_i \mapsto \zeta^i x_i$ ,  $y_i \mapsto \zeta^i y_i$ , where  $\zeta$  is a primitive fourth root of unity.

Then the quotient of  $F_{4,4}$  under the  $\mathbb{Z}/4\mathbb{Z}$  action is a smooth Godeaux surface  $Y$  with  $H_1(Y) = \mathbb{Z}/4\mathbb{Z}$  [Rei78]. For this surface and  $\sigma = 1 \in \mathbb{Z}/4\mathbb{Z}$ , the linear system  $|2K_Y + \sigma|$  has simple base points, so the condition (13) is satisfied for this particular surface. Note that  $\{q_0 = q_2 = 0\}$  define the smooth étale cover  $F_{4,4}$  of a Godeaux surface  $Y$  in  $\mathbb{P}(1, 1, 1, 2, 2)$ . The base locus of  $|2K + \sigma|$  is the union of the sets  $\{x_3 = y_1 = 0\} \cap F_{4,4}$  and  $\{x_2 = y_1 = 0\} \cap F_{4,4}$ . It is not hard to show that it consists of 16 distinct points in  $\mathbb{P}(1, 1, 1, 2, 2)$ , namely  $(1, \zeta, 0, 0, 0)$ ,  $(1, \zeta, 0, 0, -\zeta)$ ,  $(1, 0, \zeta, 0, \pm\sqrt{-\zeta - \zeta^3})$ , where  $\zeta^4 = -1$ .

Thus its quotient under the action of  $\mathbb{Z}/4\mathbb{Z}$  consists of four distinct points, so  $H^0(2K + \sigma) \otimes \mathcal{I}_P^2 = 0$ .  $\square$

Thus a vector bundle  $E$  of Theorem 3.12 is exceptional for a general Godeaux surface  $Y$  with  $\text{Tors } Y = \mathbb{Z}/4\mathbb{Z}$ . In the case  $\text{Tors } Y = \mathbb{Z}/2\mathbb{Z}$  it becomes significantly harder to write out equations for a surface. Stephen Coughlan showed by explicit calculation in Macaulay2 that condition (13) is satisfied for a least one Godeaux surface, so that it holds in general, i.e., on a Zariski open subset of the irreducible component of moduli space containing Coughlan's surface.

Note that using the Theorem 3.12 we obtain four non-isomorphic vector bundles with the same rank,  $c_1$ , and  $c_2$ , based on the choice of the point  $P$ . Since  $H^0(\mathcal{O}_Y(K_Y + \sigma) \otimes \mathcal{I}_P) = 0$ , using the exact sequence (14) we conclude that  $H^0(E) \simeq H^0(\mathcal{O}_Y) \simeq \mathbb{C}$ , thus there is exactly one section  $s \in H^0(E)$  and  $P$  is uniquely determined as the zero locus of this section.

**Corollary 3.17.** *Let  $Y$  be a general Godeaux surface with  $H_1(Y, \mathbb{Z}) = \mathbb{Z}/4\mathbb{Z}$  and  $\sigma \in \text{Tors } Y$  is an element of order 4. There exist at least four isomorphism classes of exceptional vector bundles of rank 2 on  $Y$  with  $c_1 = K_Y + \sigma$  which are stable with respect to  $K_Y$ .*

In the case  $c_1(E) \not\sim K_Y \pmod{\text{Tors } Y}$ , we provide method of constructing non-stable exceptional vector bundles of rank 2 with  $c_1(E) \not\sim K_Y \pmod{\text{Tors } Y}$ .

**Theorem 3.18.** *Let  $Z$  be a special Godeaux surface such that  $Z$  contains a  $(-3)$ -curve  $C$ . Let  $Y$  be a small deformation of  $Z$  such that  $C$  does not deform to  $Y$ . Define the line bundle  $L$  as the inverse image of the line bundle  $\mathcal{O}_Z(-C)$  under the isomorphism  $H^2(Y, \mathbb{Z}) \rightarrow H^2(Z, \mathbb{Z})$  given by specialization. Then there exists a unique extension*

$$(15) \quad 0 \rightarrow \mathcal{O}_Y \rightarrow F \rightarrow L \rightarrow 0,$$

where  $F$  is an exceptional vector bundle of rank 2 on  $Y$ , but it is not  $K_Y$ -stable.

*Proof.* By the definition of  $L$  we have  $L^2 = -3$  and  $L \cdot K_Y = -1$ . Also we can compute  $\chi(L) = 1 + (1/2)L \cdot (L - K_Y) = 0$ , and  $\chi(L^\vee) = 1 + 1/2(-L) \cdot (-L - K_Y) = -1$ .

We claim that the line bundle  $L$  satisfies  $H^0(L) = H^1(L) = H^2(L) = 0$ , and  $H^0(L^\vee) = H^2(L^\vee) = 0$ ,  $H^1(L^\vee) = \mathbb{C}$ .

Indeed,  $H^0(L) = 0$ , since  $L \cdot K_Y = -1$ , and  $K_Y$  is ample. On  $Z$  we have  $(K_Z + C) \cdot C = 1 - 3 = -2 < 0$ . Thus if  $D \in |K_Z + C|$ , then we have  $D = D' + C$ , where  $D'$  is effective. So  $D' \in |K_Z|$ , which is impossible since  $|K_Z| = \emptyset$ . Also by semicontinuity of cohomology [Har77], Ch III, Theorem 12.8, the fact that  $H^0(\mathcal{O}_Z(K_Z + C)) = 0$  on  $Z$  implies that  $H^0(\mathcal{O}_Y(K_Y) - L) = 0$  on  $Y$ . We conclude that  $H^1(L) = 0$  using the fact that  $\chi(L) = 0$ .

To show that  $H^0(L^\vee) = 0$ , note that if  $D \in H^0(L^\vee)$ , then  $D^2 = -3$ ,  $D \cdot K_Y = 1$ , so  $D$  is an irreducible curve on  $Y$ . By the adjunction formula

$$2p_a(D) - 2 = D \cdot (D + K_Y) = -2,$$

so that  $p_a(D) = 0$ . Then  $D \simeq \mathbb{P}^1$ , and by our assumption  $C$  does not deform to a  $(-3)$ -curve on  $Y$ . This is a contradiction, thus  $H^0(L^\vee) = 0$ . Finally  $H^2(L^\vee) = H^0(\mathcal{O}_Y(K_Y + L))^*$  by Serre Duality. We have  $K_Y \cdot (K_Y + L) = 0$ , but  $K_Y + L \not\sim 0$ , since for example  $L \cdot (L + K_Y) = -4 \neq 0$ , so  $H^0(\mathcal{O}_Y(K_Y + L))^* = 0$  since  $K_Y$  is ample. Finally since  $\chi(L^\vee) = -1$ , we conclude that  $H^1(L^\vee) \simeq \mathbb{C}$ .

Now we notice that the Cayley–Bacharach property (CB) is satisfied here automatically, since  $Z = \emptyset$ . We compute  $\text{Ext}^1(L, \mathcal{O}_Y) = H^1(L^\vee) \simeq \mathbb{C}$ , so the extension (15) exists, and it is unique. It is easy to see that  $F$  is not stable here, since  $c_1(F) = L$ , and  $L \cdot K_Y = -1$ .

So it only remains to show that  $F$  is exceptional. We can apply Hom from the exact sequence (15) to itself to obtain the commutative diagram shown in Figure 2.

Now  $\text{Hom}(\mathcal{O}_Y, L) = \text{Ext}^1(\mathcal{O}_Y, L) = \text{Ext}^2(\mathcal{O}_Y, L) = 0$ , since  $H^i(L) = 0$  for all  $i$ . Also  $\text{Hom}(\mathcal{O}_Y, \mathcal{O}_Y) = \mathbb{C}$ , and  $\text{Ext}^i(\mathcal{O}_Y, \mathcal{O}_Y) = 0$ , for  $i = 1, 2$ . Since  $\text{Ext}^i(F, G) = H^i(F^\vee \otimes G)$ , we conclude that  $\text{Hom}(L, L) \simeq \mathbb{C}$ , and  $\text{Ext}^i(L, L) = 0$  for  $i = 1, 2$ . Also  $\text{Hom}(L, \mathcal{O}_Y) = H^0(L^\vee) = 0$ ,  $\text{Ext}^1(L, \mathcal{O}_Y) \simeq \mathbb{C}$ , and  $\text{Ext}^2(L, \mathcal{O}_Y) = 0$ .

Now since  $\text{Ext}^2(F, \mathcal{O}_Y) = \text{Ext}^2(F, L) = 0$ , we conclude that  $\text{Ext}^2(F, F) = 0$ . Also  $\text{Ext}^1(F, L) = 0$ , and the map  $\delta \neq 0$ , since  $\delta(id) = e \in \text{Ext}^1(L, \mathcal{O}_Y) \simeq \mathbb{C}$  is the extension class [Har77], Chapter III, Exercise 6.1, and, moreover, since  $\dim \text{Ext}^1(L, \mathcal{O}_Y) = 1$ ,  $\delta$  is an isomorphism. It implies that

$$\text{Hom}(F, \mathcal{O}_Y) = \text{Ext}^1(F, \mathcal{O}_Y) = 0.$$

So  $\text{Ext}^1(F, F) = 0$ , and  $\text{Hom}(F, F) \simeq \text{Hom}(F, L) \simeq \text{Hom}(L, L) \simeq \mathbb{C}$ . Thus  $F$  is exceptional.  $\square$

$$\begin{array}{ccccc}
& 0 & & 0 & & 0 \\
& \uparrow & & \uparrow & & \uparrow \\
\overset{b_3}{\longrightarrow} \text{Ext}^2(\mathcal{O}_Y, \mathcal{O}_Y) & \longrightarrow & \text{Ext}^2(\mathcal{O}_Y, F) & \longrightarrow & \text{Ext}^2(\mathcal{O}_Y, L) \rightarrow 0 \\
& \uparrow & & \uparrow & & \uparrow \\
\overset{b_2}{\longrightarrow} \text{Ext}^2(F, \mathcal{O}_Y) & \longrightarrow & \text{Ext}^2(F, F) & \longrightarrow & \text{Ext}^2(F, L) \rightarrow 0 \\
& \uparrow & & \uparrow & & \uparrow \\
\overset{b_1}{\longrightarrow} \text{Ext}^2(L, \mathcal{O}_Y) & \longrightarrow & \text{Ext}^2(L, F) & \longrightarrow & \text{Ext}^2(L, L) \rightarrow 0 \\
& \uparrow & & \uparrow & & \uparrow \\
\overset{a_3}{\longrightarrow} \text{Ext}^1(\mathcal{O}_Y, \mathcal{O}_Y) & \longrightarrow & \text{Ext}^1(\mathcal{O}_Y, F) & \longrightarrow & \text{Ext}^1(\mathcal{O}_Y, L) \xrightarrow{b_3} \\
& \uparrow & & \uparrow & & \uparrow \\
\overset{a_2}{\longrightarrow} \text{Ext}^1(F, \mathcal{O}_Y) & \longrightarrow & \text{Ext}^1(F, F) & \longrightarrow & \text{Ext}^1(F, L) \xrightarrow{b_2} \\
& \uparrow & & \uparrow & & \uparrow \\
\overset{a_1}{\longrightarrow} \text{Ext}^1(L, \mathcal{O}_Y) & \longrightarrow & \text{Ext}^1(L, F) & \longrightarrow & \text{Ext}^1(L, L) \xrightarrow{b_1} \\
& \uparrow & & \uparrow & & \uparrow \\
& \delta & & & & \\
0 \rightarrow \text{Hom}(\mathcal{O}_Y, \mathcal{O}_Y) & \longrightarrow & \text{Hom}(\mathcal{O}_Y, F) & \longrightarrow & \text{Hom}(\mathcal{O}_Y, L) \xrightarrow{a_3} \\
& \uparrow & & \uparrow & & \uparrow \\
0 \rightarrow \text{Hom}(F, \mathcal{O}_Y) & \longrightarrow & \text{Hom}(F, F) & \longrightarrow & \text{Hom}(F, L) \xrightarrow{a_2} \\
& \uparrow & & \uparrow & & \uparrow \\
0 \rightarrow \text{Hom}(L, \mathcal{O}_Y) & \longrightarrow & \text{Hom}(L, F) & \longrightarrow & \text{Hom}(L, L) \xrightarrow{a_1} \\
& \uparrow & & \uparrow & & \uparrow \\
& 0 & & 0 & & 0
\end{array}$$

FIGURE 2. Commutative diagram obtained from applying  $\text{Hom}$  from the exact sequence (15) to itself

**Remark 3.19.** Note that the vector bundle  $F$  from Theorem 3.18 can be obtained by applying the construction of Hacking (Theorem 3.7) to birational modification of the family  $\mathcal{Z}/\Delta$ , with special fiber  $Z$  and general fiber  $Y$ .

We can blow  $Z$  up at a point of  $C$ , to obtain a surface  $\tilde{X}$  containing a chain of two  $\mathbb{P}^1$ 's with self intersections  $(-4)$ , and  $(-1)$ , intersecting at one point. Now we can contract the  $(-4)$  curve on  $\tilde{X}$  to obtain a surface  $X$  with a unique  $\frac{1}{4}(1, 1)$  singularity. Let  $\Gamma$  be the image of  $(-1)$ -curve on  $X$ . Then the birational map  $X \dashrightarrow Z$ ,  $X \setminus \Gamma \simeq Z \setminus C$  extends to a birational map  $\mathcal{X} \dashrightarrow \mathcal{Z}$  over  $\Delta$ , with  $\mathcal{X} \setminus \Gamma \simeq \mathcal{Z} \setminus C$ , where  $\mathcal{X}$  is a 3-fold with terminal singularities. Thus according to [Hac13a], we can construct

a reflexive sheaf  $\mathcal{E}$  on  $\mathcal{X}$  such that its restriction  $F$  on a nearby smooth fiber  $Y$  is exceptional, and  $sp(c_1(F)) = 2\Gamma \in H_2(X, \mathbb{Z})$ , where  $sp : H_2(Y, \mathbb{Z}) \rightarrow H_2(X, \mathbb{Z})$  is a specialization map. Then  $c_1(F) = L \in H_2(Z, \mathbb{Z})$ , and one can show that  $F$  is isomorphic to the bundle constructed in Theorem 3.18.

We now show that a degeneration  $Y \rightsquigarrow Z$  as in Theorem 3.18 exists for a Godeaux surface  $Y$  with  $\text{Tors } Y = \mathbb{Z}/5\mathbb{Z}$ .

**Theorem 3.20.** *There exists a Godeaux surface  $Y$  with  $H_1(Y, \mathbb{Z}) \simeq \mathbb{Z}/5\mathbb{Z}$ , containing no  $(-3)$ -curves.*

*Proof.* Suppose that a Godeaux surface  $Y$  with  $H_1(Y, \mathbb{Z}) = \mathbb{Z}/5\mathbb{Z}$  contains a  $(-3)$  curve  $C$ . Then by adjunction formula  $K_Y \cdot C = 1$ . On the étale cover  $\phi : Z \rightarrow Y$  of  $Y$  corresponding to  $\text{Tors } Y = \mathbb{Z}/5\mathbb{Z}$  the preimage  $\phi^{-1}(C)$  therefore has to be a union of five disjoint copies of  $(-3)$  curves, i.e. five lines on a quintic surface.

Let us show that these lines do not appear on a general  $\mathbb{Z}/5\mathbb{Z}$ -invariant quintic surface. We can list all the lines explicitly for the Fermat quintic and then check by a first order deformation theory calculation that these lines do not deform to a nearby general surface.

Let  $S = (F = 0) \subset \mathbb{P}^3$  be a Fermat quintic, and let  $L = (X = Y = 0)$  be a line on  $S$ . Then we can write  $F$  as  $F = AX + BY$  for some quartic forms  $A$  and  $B$ .

Consider a family  $\mathcal{S} \rightarrow T$ , where  $T = k[t]/(t^2)$  such that  $\mathcal{S}_0 = S$  and  $\mathcal{S}_t = (F + tG = 0) \subset \mathbb{P}^3$ , where  $G$  is  $\mathbb{Z}/5\mathbb{Z}$  invariant. Then the line  $L \subset S$  deforms to  $\mathcal{L} \subset \mathcal{S}$  over  $k[t]/(t^2)$  if and only if we can write  $(F + tG)$  as  $((A + tC)(X + tZ) + (B + tD)(Y + tW))$  for some  $C, D, Z, W$  of corresponding degrees 4, 4, 1, 1. If this is possible, then we can write  $G = AZ + CX + BW + DY$ , so  $G \in (X, Y, A, B)$ . A Macaulay calculation shows that a general  $\mathbb{Z}/5\mathbb{Z}$ -invariant  $G$  cannot be written in this form for any of the lines on the Fermat quintic. Thus there are no  $(-3)$  curves on a general Godeaux surface  $Y$  with  $H_1(Y) = \mathbb{Z}/5\mathbb{Z}$ .  $\square$

#### 4. CORRESPONDENCE

This section summarizes our results relating to the correspondence (1) between degenerations and vector bundles.

**Theorem 4.1.** *Let  $Y \rightsquigarrow X$  be a  $\mathbb{Q}$ -Gorenstein degeneration, where  $Y$  is a Godeaux surface, and  $X$  has a unique singularity of type  $\frac{1}{4}(1, 1)$  and  $K_X$  is ample. Assume  $H_1(Y, \mathbb{Z}) \simeq H_1(X, \mathbb{Z})$ . Let  $\sigma \in H_1(Y)$ . The construction of Hacking produces an exceptional vector bundle  $E$  of rank 2 on  $Y$  with  $c_1(E) = K_Y + \sigma$  modulo the equivalence relation if and only if  $K_X + \sigma \in H_2(Y, \mathbb{Z})$  is a 2-divisible divisor on  $X$ .*

*Proof.* Suppose  $K_X + \sigma$  is 2-divisible where  $\sigma \in H_1(X) = \text{Tors } H^2(X) = \text{Tors Pic } X$  is torsion. Write  $K_X + \sigma = 2D$ .

The local class group of the singularity  $P \in X \simeq \frac{1}{4}(1, 1)$  is isomorphic to  $\mathbb{Z}/4\mathbb{Z}$ . Here  $w \in \mathbb{Z}/4\mathbb{Z}$  corresponds to the class of a divisor given by an equation  $(f(u, v) = 0) \subset \mathbb{C}_{u,v}^2 / \frac{1}{4}(1, 1)$  where  $f$  has weight  $w$  with respect to the group action, i.e., under the generator  $(u, v) \mapsto (\zeta u, \zeta v)$  we have  $f \mapsto \zeta^w f$ . Note that the local class of  $K_X + \sigma$  corresponds to  $2 \in \mathbb{Z}/4\mathbb{Z}$ . (Indeed, a local section of  $\omega_X = \mathcal{O}_X(K_X)$  is given by  $\Omega = f(u, v)du \wedge dv$ , where  $f$  has weight 2 (so that  $\Omega$  is invariant with respect to the group action). Then its zero locus  $(\Omega = 0) = (f(u, v) = 0)$  is a divisor in the class  $K_X$ .) Also  $\sigma$  is a Cartier divisor (i.e., corresponds to a line bundle) so

$\sigma = 0$  in the local class group. Thus  $D$  corresponds to  $\pm 1 \in \mathbb{Z}/4\mathbb{Z}$ , equivalently,  $\pm D$  is locally linearly equivalent to  $(v = 0)$ . Now by [Hac13a], Proposition 4.2, we obtain an exceptional bundle  $F$  on  $Y$  with  $c_1(F) = \pm 2D = \pm(K_Y + \sigma)$ . Replacing  $F$  by  $F^*$  if necessary we obtain  $c_1(F) = K_Y + \sigma$ .

Conversely, suppose  $F$  is an exceptional bundle with  $c_1(F) = K_Y + \sigma$  associated to a degeneration. Then we have  $c_1(F) = 2D \in H_2(X, \mathbb{Z}) = \text{Cl}(X)$  for some divisor  $D$  on  $X$  by [Hac13a], Theorem 1.1  $\square$

**Proposition 4.2.** *Let  $Y \rightsquigarrow X$  be a degeneration of a smooth Godeaux surface  $Y$  to a surface  $X$  with a unique singularity  $(P \in X)$  of Wahl type  $\frac{1}{4}(1, 1)$ . Assume that  $H_1(Y) \simeq H_1(X)$ . Then*

$$\text{Tors } H_2(X) \simeq \text{Tors } H^2(X) \simeq \text{Tors } H^2(Y).$$

*Proof.* Write  $X = X^0 \cup_L C$ , where  $L$  is the link of singularity, and  $C$  is the cone of singularity, and  $X^0 = X \setminus C$ . Then there is a Mayer–Vietoris sequence

$$H_2(X) \rightarrow H_1(L) \rightarrow H_1(X^0) \oplus H_1(C) \rightarrow H_1(X) \rightarrow 0.$$

Since  $H_1(Y) \simeq H_1(X)$ , by [Hac13b, p. 134], we obtain  $H_2(X) \rightarrow H_1(L)$ , thus  $H_1(X^0) \simeq H_1(X)$  since  $C$  is contractible.

By the Universal coefficient theorem  $\text{Tors } H^2(X^0) \simeq \text{Tors } H_1(X^0) = H_1(X)$  since  $H_1(X^0) = H_1(X)$  is torsion, and  $\text{Tors } H^2(Y) \simeq \text{Tors } H_1(Y) = H_1(Y)$ .

Now by Poincaré Duality  $\text{Tors } H_2(X) = \text{Tors } H^2(X^0)$ . So  $\text{Tors } H_2(X) \simeq H_1(X)$ . Thus  $\text{Tors } H_2(X) \simeq \text{Tors } H^2(Y)$ .  $\square$

**Proposition 4.3.** *In the classification of the minimal resolutions  $\tilde{X}$  of the  $\mathbb{Q}$ -Gorenstein degenerations  $Y \rightsquigarrow X$  in Theorem 2.3 the canonical class  $K_X + \sigma$  is 2-divisible in  $H_2(X)$  for some  $\sigma \in \text{Tors } X$  precisely in the cases (a) and (c).*

*Proof.* We have  $\text{Tors } H_2(X, \mathbb{Z}) \simeq \text{Tors } H^2(Y, \mathbb{Z})$ . So  $K_X + \sigma \in H_2(X)$  is 2-divisible for some  $\sigma \in \text{Tors } H_2(X) = \text{Tors } H^2(Y)$  if and only if  $K_X \in H_2(X)/\text{Tors } H_2(X)$  is 2-divisible.

Let  $E \simeq \mathbb{P}^1$  be the exceptional locus of the minimal resolution  $\pi : \tilde{X} \rightarrow X$  of  $X$ . Then  $\tilde{X} = X^0 \cup N$ , where  $N = \pi^{-1}C$  is homotopy equivalent to  $E \simeq \mathbb{P}^1$ . We have an exact sequence

$$(16) \quad 0 \rightarrow \mathbb{Z} \rightarrow H_2(\tilde{X}, \mathbb{Z}) \rightarrow H_2(X, \mathbb{Z}) \rightarrow 0,$$

where the first map is given by  $1 \mapsto [E]$ . Since  $\text{Tors } H_2(\tilde{X}) = \text{Tors } H_2(X)$ , the exact sequence (16) is split, i.e.  $H_2(\tilde{X}, \mathbb{Z}) \simeq \mathbb{Z} \cdot [E] \oplus H_2(X, \mathbb{Z})$ , defined by  $\alpha \mapsto (\theta(\alpha) \cdot [E] \oplus \pi_* \alpha)$  for some  $\theta : H_2(\tilde{X}) \rightarrow \mathbb{Z}$ .

Now  $\pi_* K_{\tilde{X}} = K_X$ . So  $K_X$  is 2-divisible in  $H_2(X, \mathbb{Z})/\text{Tors } H_2(X, \mathbb{Z})$  if and only if  $K_{\tilde{X}}$  or  $K_{\tilde{X}} + E$  is 2-divisible in  $H_2(\tilde{X}, \mathbb{Z})$ .

Note that  $K_{\tilde{X}} + E$  is not 2-divisible in  $H_2(\tilde{X}, \mathbb{Z})$ . Indeed, if  $K_{\tilde{X}} + E = 2D$  for some  $D \in H_2(\tilde{X}, \mathbb{Z})$ , then  $K_{\tilde{X}} \cdot D = 1$  and  $D^2 = 0$ , which is a contradiction since we always have  $K_{\tilde{X}} \cdot D \equiv D^2 \pmod{2}$ .

So it only remains to check when  $K_{\tilde{X}}$  is divisible in  $H_2(\tilde{X}, \mathbb{Z})/\text{Tors } H_2(\tilde{X}, \mathbb{Z})$ .

Let  $K_{\tilde{X}} = \lambda A$ , where  $A$  is a general fiber of the elliptic fibration  $\tilde{X} \rightarrow \mathbb{P}^1$ . Let  $m = \text{lcm}(m_1, m_2)$ . By the Kodaira Canonical Bundle formula (2) we have

$$\lambda = 1 - \frac{1}{m_1} - \frac{1}{m_2}.$$

Clearly, if  $\mu \in \mathbb{Q}$ ,  $\mu m \in \mathbb{Z}$ , then  $\mu A \in H_2(\tilde{X}, \mathbb{Z})$ .

Conversely, by [FM94], Chapter II, Proposition 2.7, there exists some  $D \in H_2(\tilde{X}, \mathbb{Z})$  such that  $D \cdot A = m$ . Thus if  $\mu A \in H_2(\tilde{X}, \mathbb{Z})$ , then  $D \cdot \mu A = \mu m$  is an integer. So  $K_{\tilde{X}}$  is 2-divisible in  $H_2(\tilde{X}, \mathbb{Z})/\text{Tors } H_2(\tilde{X}, \mathbb{Z})$  if and only if  $\lambda m \in 2\mathbb{Z}$ .

We have:

- (a)  $m_1 = 4, m_2 = 4, \lambda m = 2, K_{\tilde{X}} = \frac{1}{2}A = 2F_4$ ;
- (b)  $m_1 = 3, m_2 = 3, \lambda m = 1, K_{\tilde{X}} = \frac{1}{3}A = F_3$ ;
- (c)  $m_1 = 2, m_2 = 6, \lambda m = 2, K_{\tilde{X}} = \frac{1}{3}A = 2F_6$ ;
- (d)  $m_1 = 2, m_2 = 4, \lambda m = 1, K_{\tilde{X}} = \frac{1}{4}A = F_4$ ;
- (e)  $m_1 = 2, m_2 = 3, \lambda m = 1, K_{\tilde{X}} = \frac{1}{6}A = F_2 - F_3$ .

Thus  $K_X$  is 2-divisible in  $H_2(X)/\text{Tors } X$  in cases (a) and (c).  $\square$

*Proof of the Theorem 1.4.* In the case  $H_1(Y, \mathbb{Z}) = \mathbb{Z}/4\mathbb{Z}$ , a degeneration  $Y \rightsquigarrow X$ , where  $X$  has a unique Wahl singularity of type  $\frac{1}{4}(1, 1)$  is explicitly constructed in Proposition 2.6. By Proposition 4.3,  $K_X$  is 2-divisible in  $H_2(X)/\text{Tors } X$ . According to the Theorem 4.1, we can produce an exceptional vector bundle  $E$  of rank 2 on  $Y$  with  $c_1(E) = K_Y + \sigma$  using the construction described in Theorem 3.7. Every such vector bundle is equivalent to the vector bundle defined in Theorem 3.12.  $\square$

We were not able to construct exceptional vector bundles  $E$  of rank 2 on  $Y$  such that  $c_1(E) \not\sim K_Y$ , and  $E$  is stable with respect to  $K_Y$  directly. The construction of Hacking guarantees the existence of such vector bundles in the cases  $H_1(Y, \mathbb{Z}) = \mathbb{Z}/3\mathbb{Z}$ ,  $H_1(Y, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$  or  $H_1(Y, \mathbb{Z}) = 0$ .

The main open problem is to determine whether there are exceptional vector bundles of rank 2 on  $Y$  such that  $c_1(E) \not\sim K_Y$ , and such that  $E$  is stable with respect to  $K_Y$  in the cases  $H_1(Y, \mathbb{Z}) = \mathbb{Z}/4\mathbb{Z}$  and  $H_1(Y, \mathbb{Z}) = \mathbb{Z}/5\mathbb{Z}$ . If such vector bundles exist, then they cannot possibly come from degenerations, because no such boundary components exist in the classification of Theorem 2.3, and Proposition 4.3. Interestingly, in the case  $H_1(Y) = \mathbb{Z}/5\mathbb{Z}$  there do exist exceptional vector bundles of rank 2 which are not  $K_Y$ -stable, and moreover these can be obtained from degenerations to a surface  $X$  with  $\frac{1}{4}(1, 1)$  singularity for which  $K_X$  is not nef, as shown in Theorem 3.18.

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